

THE SUPER $\mathcal{W}_{1+\infty}$ ALGEBRA WITH INTEGRAL CENTRAL CHARGE

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ABSTRACT. The Lie superalgebra \mathcal{SD} of regular differential operators on the super circle has a universal central extension $\widehat{\mathcal{SD}}$. For each $c \in \mathbb{C}$, the vacuum module $\mathcal{M}_c(\widehat{\mathcal{SD}})$ of central charge c admits a vertex superalgebra structure, and $\mathcal{M}_c(\widehat{\mathcal{SD}}) \cong \mathcal{M}_{-c}(\widehat{\mathcal{SD}})$. The irreducible quotient $\mathcal{V}_c(\widehat{\mathcal{SD}})$ of the vacuum module is known as the super $\mathcal{W}_{1+\infty}$ algebra. We show that for each integer $n > 0$, $\mathcal{V}_n(\widehat{\mathcal{SD}})$ has a minimal strong generating set consisting of $4n$ fields, and we identify it with a \mathcal{W} -algebra associated to the purely odd simple root system of $\mathfrak{gl}(n|n)$. Finally, we realize $\mathcal{V}_n(\widehat{\mathcal{SD}})$ as the limit of a family of commutant vertex algebras that generically have the same graded character and possess a minimal strong generating set of the same cardinality.

1. INTRODUCTION

Let \mathcal{D} denote the Lie algebra of regular differential operators on the circle. It has a universal central extension $\hat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}\kappa$ which was introduced by Kac-Peterson in [KP]. Although $\hat{\mathcal{D}}$ admits a principal \mathbb{Z} -gradation and triangular decomposition, its representation theory is nontrivial because the graded pieces are all infinite-dimensional. The important problem of constructing and classifying the *quasifinite* irreducible, highest-weight representations (i.e., those with finite-dimensional graded pieces) was solved by Kac-Radul in [KRI]. Explicit constructions of these modules were given in terms of the representation theory of $\widehat{\mathfrak{gl}}(\infty, R_m)$, which is a central extension of the Lie algebra of infinite matrices over $R_m = \mathbb{C}[t]/(t^{m+1})$ having only finitely many nonzero diagonal elements. The authors also classified all such $\hat{\mathcal{D}}$ -modules which are unitary.

In [FKRW], the representation theory of $\hat{\mathcal{D}}$ was developed from the point of view of vertex algebras. For each $c \in \mathbb{C}$, $\hat{\mathcal{D}}$ admits a module \mathcal{M}_c called the *vacuum module*, which is a vertex algebra freely generated by fields J^l of weight $l + 1$, for $l \geq 0$. The highest-weight representations of $\hat{\mathcal{D}}$ are in one-to-one correspondence with the highest-weight representations of \mathcal{M}_c . The irreducible quotient of \mathcal{M}_c by its maximal graded, proper $\hat{\mathcal{D}}$ -submodule \mathcal{I}_c is a simple vertex algebra, and is often denoted by $\mathcal{W}_{1+\infty, c}$. These algebras have been studied extensively in both the physics and mathematics literature (see for example [AFMO][ASV][BS][CTZ][FKRW][KRII]), and they play an important role in the theory of integrable systems. The above central extension is normalized so that \mathcal{M}_c is reducible if and only if $c \in \mathbb{Z}$. It was shown in [FKRW] that for every integer $n \geq 1$, \mathcal{I}_n is generated as a vertex algebra ideal by a singular vector of weight $n + 1$, and

$$(1.1) \quad \mathcal{W}_{1+\infty, n} \cong \mathcal{W}(\mathfrak{gl}_n).$$

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In particular, $\mathcal{W}_{1+\infty,n}$ has a minimal strong generating set consisting of a field in each weight $1, 2, \dots, n$. The case of negative integral central charge is more complicated. It was shown in [LI] that \mathcal{I}_{-n} is generated by a singular vector of weight $(n+1)^2$, and $\mathcal{W}_{1+\infty,-n}$ has a minimal strong generating set consisting of a field in each weight $1, 2, \dots, n^2 + 2n$. Wang showed in [W] that $\mathcal{W}_{1+\infty,-1}$ is isomorphic to $\mathcal{W}(\mathfrak{gl}_3)$ with central charge -2 , but for $n > 1$ it is not known if $\mathcal{W}_{1+\infty,-n}$ can be identified with a standard \mathcal{W} -algebra.

The super analogue of \mathcal{D} is the Lie superalgebra \mathcal{SD} of regular differential operators on the super circle $S^{1|1}$. As above, it has a universal central extension $\widehat{\mathcal{SD}}$, and for each $c \in \mathbb{C}$ $\widehat{\mathcal{SD}}$ admits a vacuum module $\mathcal{M}_c(\widehat{\mathcal{SD}})$. This module has a vertex superalgebra structure, and is freely generated by fields

$$\{J^{0,k}, J^{1,k}, J^{+,k}, J^{-,k} \mid k \geq 0\}$$

of weights $k+1, k+1, k+1/2, k+3/2$, respectively. Unlike the modules \mathcal{M}_c which are all distinct, $\mathcal{M}_c(\widehat{\mathcal{SD}}) \cong \mathcal{M}_{-c}(\widehat{\mathcal{SD}})$ for all c . There are actions of the affine vertex superalgebra $V_c(\mathfrak{gl}(1|1))$ and the $N = 2$ superconformal algebra \mathcal{A}_c of central charge c on $\mathcal{M}_c(\widehat{\mathcal{SD}})$. Moreover, $\{J^{0,k} \mid k \geq 0\}$ and $\{J^{1,k} \mid k \geq 0\}$ generate copies of \mathcal{M}_c and \mathcal{M}_{-c} , respectively, which form a Howe pair (i.e., a pair of mutual commutants) inside $\mathcal{M}_c(\widehat{\mathcal{SD}})$.

The super $\mathcal{W}_{1+\infty}$ algebra $\mathcal{V}_c(\widehat{\mathcal{SD}})$ is the unique irreducible quotient of $\mathcal{M}_c(\widehat{\mathcal{SD}})$ by its maximal proper graded $\widehat{\mathcal{SD}}$ -submodule \mathcal{SI}_c . We denote the map $\mathcal{M}_c(\widehat{\mathcal{SD}}) \rightarrow \mathcal{V}_c(\widehat{\mathcal{SD}})$ by π_c , and we denote $\pi_c(J^{a,k})$ by $j^{a,k}$ for $a = 0, 1, \pm$. There are induced actions of $V_c(\mathfrak{gl}(1|1))$ and \mathcal{A}_c on $\mathcal{V}_c(\widehat{\mathcal{SD}})$, as well as copies of $\mathcal{W}_{1+\infty,c}$ and $\mathcal{W}_{1+\infty,-c}$ which form a Howe pair inside $\mathcal{V}_c(\widehat{\mathcal{SD}})$.

For $n \in \mathbb{Z}$, $\mathcal{M}_n(\widehat{\mathcal{SD}})$ is reducible and $\mathcal{V}_n(\widehat{\mathcal{SD}})$ has a nontrivial structure. Our main goal in this paper is to elucidate this structure. Since $\mathcal{V}_n(\widehat{\mathcal{SD}}) \cong \mathcal{V}_{-n}(\widehat{\mathcal{SD}})$ and $\mathcal{V}_0(\widehat{\mathcal{SD}}) \cong \mathbb{C}$, it suffices to consider the case $n \geq 1$. This problem was posed by Cheng-Wang; see Problem 3 at the end of [CW]. Our starting point is a free field realization of $\mathcal{V}_n(\widehat{\mathcal{SD}})$ due to Awata-Fukuma-Matsuo-Otake as the GL_n -invariant subalgebra of the $bc\beta\gamma$ -system \mathcal{F} of rank n [AFMO]. We will show that \mathcal{SI}_n is generated as a vertex algebra ideal by a singular vector of weight $n+1/2$, and that $\mathcal{V}_n(\widehat{\mathcal{SD}})$ has a minimal strong generating set consisting of the following $4n$ fields:

$$\{j^{0,k}, j^{1,k}, j^{+,k}, j^{-,k} \mid k = 0, \dots, n-1\}.$$

Next we show that $\mathcal{V}_n(\widehat{\mathcal{SD}})$ admits a deformation as the limit of a family of commutant algebras. The rank n $bc\beta\gamma$ -system \mathcal{F} has a natural action of $V_0(\mathfrak{gl}_n)$, and we obtain a diagonal homomorphism $V_k(\mathfrak{gl}_n) \rightarrow V_k(\mathfrak{gl}_n) \otimes \mathcal{F}$ for all k . We define

$$\mathcal{B}_{n,k} = \text{Com}(V_k(\mathfrak{gl}_n), V_k(\mathfrak{gl}_n) \otimes \mathcal{F}).$$

We have $\lim_{k \rightarrow \infty} \mathcal{B}_{n,k} = \mathcal{V}_n(\widehat{\mathcal{SD}})$, and for generic values of k , $\mathcal{B}_{n,k}$ has a minimal strong generating set consisting of $4n$ generators and has the same graded character as $\mathcal{V}_n(\widehat{\mathcal{SD}})$.

Next, we consider a family of \mathcal{W} -algebras $\mathcal{W}_{n,k}$ associated naturally to the Lie superalgebra $\mathfrak{gl}(n|n)$. It is defined as a certain subalgebra of the joint kernel of screening operators corresponding to the purely odd simple root system of $\mathfrak{gl}(n|n)$. We expect that $\mathcal{W}_{n,k}$ coincides with the joint kernel of the screening charges, although we are unable to prove this at present. The \mathcal{W} -algebras of simple affine Lie (super) algebras $\hat{\mathfrak{g}}$ [FF1, FF2, KRW]

are defined via the quantum Hamiltonian reduction, which is a certain semi-infinite cohomology. These \mathcal{W} -algebras are associated to the principal embedding of \mathfrak{sl}_2 in \mathfrak{g} and they usually can also be realized as the intersection of kernels of screening charges corresponding to a simple root system of \mathfrak{g} . A simple root system of a Lie superalgebra is not unique, and in our case it turns out that a purely odd simple root system is most suitable. We will show that

$$(1.2) \quad \mathcal{V}_n(\widehat{\mathcal{SD}}) \cong \lim_{k \rightarrow \infty} \mathcal{W}_{n,k},$$

and we regard this as an analogue of the isomorphism $\mathcal{W}_{1+\infty,n} \cong \mathcal{W}(\mathfrak{gl}_n)$. In the case $n = 2$, we find a minimal strong generating set for $\mathcal{W}_{2,k}$ consisting of eight fields, and show by explicit computation that $\mathcal{W}_{2,k+2}$ has the same operator product algebra as $\mathcal{B}_{2,k}$. More generally, we conjecture that $\mathcal{W}_{n,k+n}$ is isomorphic to $\mathcal{B}_{n,k}$ for all k and n .

There is also a commutant realization of the deformable family of \mathcal{W} -algebras of \mathfrak{sl}_n , namely $\text{Com}(V_{k+1}(\mathfrak{sl}_n), V_k(\mathfrak{sl}_n) \otimes V_1(\mathfrak{sl}_n))$ [BS]. In physics the corresponding conformal field theories are called \mathcal{W}_n minimal models and they have received much attention recently as tentative dual theories to three dimensional higher spin gravity [GG]. The supersymmetric analogue [CHR] has the \mathcal{W} -superalgebra of $\mathfrak{sl}(n+1|n)$ as coset algebra whose twisted algebra in turn is argued to be related to the \mathcal{W} -superalgebra of $\mathfrak{gl}(n|n)$ [I].

2. VERTEX ALGEBRAS

In this section, we define vertex algebras, which have been discussed from various different points of view in the literature [B][FBZ][FHL][FLM][K][LiI][LZ]. We will follow the formalism developed in [LZ] and partly in [LiI]. Let $V = V_0 \oplus V_1$ be a super vector space over \mathbb{C} , and let z, w be formal variables. By $\text{QO}(V)$, we mean the space of all linear maps

$$V \rightarrow V((z)) := \left\{ \sum_{n \in \mathbb{Z}} v(n) z^{-n-1} \mid v(n) \in V, v(n) = 0 \text{ for } n \gg 0 \right\}.$$

Each element $a \in \text{QO}(V)$ can be uniquely represented as a power series

$$a = a(z) := \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in \text{End}(V)[[z, z^{-1}]].$$

We refer to $a(n)$ as the n th Fourier mode of $a(z)$. Each $a \in \text{QO}(V)$ is of the shape $a = a_0 + a_1$ where $a_i : V_j \rightarrow V_{i+j}((z))$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$, and we write $|a_i| = i$.

On $\text{QO}(V)$ there is a set of nonassociative bilinear operations \circ_n , indexed by $n \in \mathbb{Z}$, which we call the n th circle products. For homogeneous $a, b \in \text{QO}(V)$, they are defined by

$$a(w) \circ_n b(w) = \text{Res}_z a(z) b(w) \iota_{|z| > |w|} (z - w)^n - (-1)^{|a||b|} \text{Res}_z b(w) a(z) \iota_{|w| > |z|} (z - w)^n.$$

Here $\iota_{|z| > |w|} f(z, w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$ denotes the power series expansion of a rational function f in the region $|z| > |w|$. We usually omit the symbol $\iota_{|z| > |w|}$ and just write $(z - w)^{-1}$ to mean the expansion in the region $|z| > |w|$, and write $-(w - z)^{-1}$ to mean the expansion in $|w| > |z|$. It is easy to check that $a(w) \circ_n b(w)$ above is a well-defined element of $\text{QO}(V)$.

The nonnegative circle products are connected through the *operator product expansion* (OPE) formula. For $a, b \in \text{QO}(V)$, we have

$$(2.1) \quad a(z)b(w) = \sum_{n \geq 0} a(w) \circ_n b(w) (z-w)^{-n-1} + :a(z)b(w):,$$

which is often written as $a(z)b(w) \sim \sum_{n \geq 0} a(w) \circ_n b(w) (z-w)^{-n-1}$, where \sim means equal modulo the term

$$:a(z)b(w): = a(z)_- b(w) + (-1)^{|a||b|} b(w) a(z)_+.$$

Here $a(z)_- = \sum_{n < 0} a(n)z^{-n-1}$ and $a(z)_+ = \sum_{n \geq 0} a(n)z^{-n-1}$. Note that $:a(w)b(w):$ is a well-defined element of $\text{QO}(V)$. It is called the *Wick product* of a and b , and it coincides with $a \circ_{-1} b$. The other negative circle products are related to this by

$$n! a(z) \circ_{-n-1} b(z) = :(\partial^n a(z))b(z):,$$

where ∂ denotes the formal differentiation operator $\frac{d}{dz}$. For $a_1(z), \dots, a_k(z) \in \text{QO}(V)$, the k -fold iterated Wick product is defined to be

$$(2.2) \quad :a_1(z)a_2(z) \cdots a_k(z): = :a_1(z)b(z):,$$

where $b(z) = :a_2(z) \cdots a_k(z):$. We often omit the formal variable z when no confusion can arise.

The set $\text{QO}(V)$ is a nonassociative algebra with the operations \circ_n , which satisfy $1 \circ_n a = \delta_{n,-1}a$ for all n , and $a \circ_n 1 = \delta_{n,-1}a$ for $n \geq -1$. In particular, 1 behaves as a unit with respect to \circ_{-1} . A linear subspace $\mathcal{A} \subset \text{QO}(V)$ containing 1 which is closed under the circle products will be called a *quantum operator algebra* (QOA). Note that \mathcal{A} is closed under ∂ since $\partial a = a \circ_{-2} 1$. Many formal algebraic notions are immediately clear: a homomorphism is just a linear map that sends 1 to 1 and preserves all circle products; a module over \mathcal{A} is a vector space M equipped with a homomorphism $\mathcal{A} \rightarrow \text{QO}(M)$, etc. A subset $S = \{a_i \mid i \in I\}$ of \mathcal{A} is said to generate \mathcal{A} if every element $a \in \mathcal{A}$ can be written as a linear combination of nonassociative words in the letters a_i, \circ_n , for $i \in I$ and $n \in \mathbb{Z}$. We say that S *strongly generates* \mathcal{A} if every $a \in \mathcal{A}$ can be written as a linear combination of words in the letters a_i, \circ_n for $n < 0$. Equivalently, \mathcal{A} is spanned by the collection $\{:\partial^{k_1} a_{i_1}(z) \cdots \partial^{k_m} a_{i_m}(z): \mid i_1, \dots, i_m \in I, k_1, \dots, k_m \geq 0\}$.

We say that $a, b \in \text{QO}(V)$ *quantum commute* if $(z-w)^N [a(z), b(w)] = 0$ for some $N \geq 0$. Here $[\cdot, \cdot]$ denotes the super bracket. This condition implies that $a \circ_n b = 0$ for $n \geq N$, so (2.1) becomes a finite sum. A *commutative quantum operator algebra* (CQOA) is a QOA whose elements pairwise quantum commute. Finally, the notion of a CQOA is equivalent to the notion of a vertex algebra. Every CQOA \mathcal{A} is itself a faithful \mathcal{A} -module, called the *left regular module*. Define

$$\rho : \mathcal{A} \rightarrow \text{QO}(\mathcal{A}), \quad a \mapsto \hat{a}, \quad \hat{a}(\zeta)b = \sum_{n \in \mathbb{Z}} (a \circ_n b) \zeta^{-n-1}.$$

Then ρ is an injective QOA homomorphism, and the quadruple of structures $(\mathcal{A}, \rho, 1, \partial)$ is a vertex algebra in the sense of [FLM]. Conversely, if $(V, Y, 1, D)$ is a vertex algebra, the collection $Y(V) \subset \text{QO}(V)$ is a CQOA. We will refer to a CQOA simply as a vertex algebra throughout the rest of this paper.

Let \mathcal{R} be the category of vertex algebras \mathcal{A} equipped with a $\mathbb{Z}_{\geq 0}$ -filtration

$$(2.3) \quad \mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots, \quad \mathcal{A} = \bigcup_{k \geq 0} \mathcal{A}_{(k)}$$

such that $\mathcal{A}_{(0)} = \mathbb{C}$, and for all $a \in \mathcal{A}_{(k)}$, $b \in \mathcal{A}_{(l)}$, we have

$$(2.4) \quad a \circ_n b \in \mathcal{A}_{(k+l)}, \quad \text{for } n < 0,$$

$$(2.5) \quad a \circ_n b \in \mathcal{A}_{(k+l-1)}, \quad \text{for } n \geq 0.$$

Elements $a(z) \in \mathcal{A}_{(d)} \setminus \mathcal{A}_{(d-1)}$ are said to have degree d .

Filtrations on vertex algebras satisfying (2.4)-(2.5) were introduced in [LiII], and are known as *good increasing filtrations*. Setting $\mathcal{A}_{(-1)} = \{0\}$, the associated graded object $\text{gr}(\mathcal{A}) = \bigoplus_{k \geq 0} \mathcal{A}_{(k)} / \mathcal{A}_{(k-1)}$ is a $\mathbb{Z}_{\geq 0}$ -graded associative, (super)commutative algebra with a unit 1 under a product induced by the Wick product on \mathcal{A} . For each $r \geq 1$ we have the projection

$$(2.6) \quad \phi_r : \mathcal{A}_{(r)} \rightarrow \mathcal{A}_{(r)} / \mathcal{A}_{(r-1)} \subset \text{gr}(\mathcal{A}).$$

Moreover, $\text{gr}(\mathcal{A})$ has a derivation ∂ of degree zero (induced by the operator $\partial = \frac{d}{dz}$ on \mathcal{A}), and for each $a \in \mathcal{A}_{(d)}$ and $n \geq 0$, the operator $a \circ_n$ on \mathcal{A} induces a derivation of degree $d - k$ on $\text{gr}(\mathcal{A})$, which we denote by $a(n)$. Here

$$k = \sup\{j \geq 1 \mid \mathcal{A}_{(r)} \circ_n \mathcal{A}_{(s)} \subset \mathcal{A}_{(r+s-j)}, \forall r, s, n \geq 0\},$$

as in [LL]. Finally, these derivations give $\text{gr}(\mathcal{A})$ the structure of a vertex Poisson algebra.

The assignment $\mathcal{A} \mapsto \text{gr}(\mathcal{A})$ is a functor from \mathcal{R} to the category of $\mathbb{Z}_{\geq 0}$ -graded (super)commutative rings with a differential ∂ of degree zero, which we will call ∂ -rings. A ∂ -ring is just an *abelian* vertex algebra, that is, a vertex algebra \mathcal{V} in which $[a(z), b(w)] = 0$ for all $a, b \in \mathcal{V}$. A ∂ -ring A is said to be generated by a subset $\{a_i \mid i \in I\}$ if $\{\partial^k a_i \mid i \in I, k \geq 0\}$ generates A as a graded ring. The key feature of \mathcal{R} is the following reconstruction property [LL]:

Lemma 2.1. *Let \mathcal{A} be a vertex algebra in \mathcal{R} and let $\{a_i \mid i \in I\}$ be a set of generators for $\text{gr}(\mathcal{A})$ as a ∂ -ring, where a_i is homogeneous of degree d_i . If $a_i(z) \in \mathcal{A}_{(d_i)}$ are vertex operators such that $\phi_{d_i}(a_i(z)) = a_i$, then \mathcal{A} is strongly generated as a vertex algebra by $\{a_i(z) \mid i \in I\}$.*

As shown in [LI], there is a similar reconstruction property for kernels of surjective morphisms in \mathcal{R} . Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism in \mathcal{R} with kernel \mathcal{J} , such that f maps $\mathcal{A}_{(k)}$ onto $\mathcal{B}_{(k)}$ for all $k \geq 0$. The kernel J of the induced map $\text{gr}(f) : \text{gr}(\mathcal{A}) \rightarrow \text{gr}(\mathcal{B})$ is a homogeneous ∂ -ideal (i.e., $\partial J \subset J$). A set $\{a_i \mid i \in I\}$ such that a_i is homogeneous of degree d_i is said to generate J as a ∂ -ideal if $\{\partial^k a_i \mid i \in I, k \geq 0\}$ generates J as an ideal.

Lemma 2.2. *Let $\{a_i \mid i \in I\}$ be a generating set for J as a ∂ -ideal, where a_i is homogeneous of degree d_i . Then there exist vertex operators $a_i(z) \in \mathcal{A}_{(d_i)}$ with $\phi_{d_i}(a_i(z)) = a_i$, such that $\{a_i(z) \mid i \in I\}$ generates \mathcal{J} as a vertex algebra ideal.*

3. THE $\mathcal{W}_{1+\infty}$ ALGEBRA

Let \mathcal{D} be the Lie algebra of regular differential operators on $\mathbb{C} \setminus \{0\}$, with coordinate t . A standard basis for \mathcal{D} is

$$J_k^l = -t^{l+k}(\partial_t)^l, \quad k \in \mathbb{Z}, \quad l \in \mathbb{Z}_{\geq 0},$$

where $\partial_t = \frac{d}{dt}$. An alternative basis is $\{t^k D^l \mid k \in \mathbb{Z}, l \in \mathbb{Z}_{\geq 0}\}$, where $D = t\partial_t$. There is a 2-cocycle on \mathcal{D} given by

$$(3.1) \quad \Psi\left(f(t)(\partial_t)^m, g(t)(\partial_t)^n\right) = \frac{m!n!}{(m+n+1)!} \text{Res}_{t=0} f^{(n+1)}(t)g^{(m)}(t)dt,$$

and a corresponding central extension $\hat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}\kappa$, which was first studied by Kac-Peterson in [KP]. $\hat{\mathcal{D}}$ has a \mathbb{Z} -grading $\hat{\mathcal{D}} = \bigoplus_{j \in \mathbb{Z}} \hat{\mathcal{D}}_j$ by weight, given by

$$\text{wt}(J_k^l) = k, \quad \text{wt}(\kappa) = 0,$$

and a triangular decomposition $\hat{\mathcal{D}} = \hat{\mathcal{D}}_+ \oplus \hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_-$, where $\hat{\mathcal{D}}_{\pm} = \bigoplus_{j \in \pm\mathbb{N}} \hat{\mathcal{D}}_j$ and $\hat{\mathcal{D}}_0 = \mathcal{D}_0 \oplus \mathbb{C}\kappa$.

Let \mathcal{P} be the parabolic subalgebra of \mathcal{D} consisting of differential operators which extend to all of \mathbb{C} , which has a basis $\{J_k^l \mid l \geq 0, l+k \geq 0\}$. The cocycle Ψ vanishes on \mathcal{P} , so \mathcal{P} may be regarded as a subalgebra of $\hat{\mathcal{D}}$, and $\hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_+ \subset \hat{\mathcal{P}}$, where $\hat{\mathcal{P}} = \mathcal{P} \oplus \mathbb{C}\kappa$. Given $c \in \mathbb{C}$, let \mathbb{C}_c denote the one-dimensional $\hat{\mathcal{P}}$ -module on which κ acts by $c \cdot \text{id}$ and J_k^l acts by zero. The induced $\hat{\mathcal{D}}$ -module

$$\mathcal{M}_c = U(\hat{\mathcal{D}}) \otimes_{U(\hat{\mathcal{P}})} \mathbb{C}_c$$

is known as the *vacuum $\hat{\mathcal{D}}$ -module of central charge c* . \mathcal{M}_c has a vertex algebra structure and is generated by fields

$$J^l(z) = \sum_{k \in \mathbb{Z}} J_k^l z^{-k-l-1}, \quad l \geq 0$$

of weight $l+1$. The modes J_k^l represent $\hat{\mathcal{D}}$ on \mathcal{M}_c , and as in [LI], we rewrite these fields in the form

$$J^l(z) = \sum_{k \in \mathbb{Z}} J^l(k) z^{-k-1},$$

where $J^l(k) = J_{k-l}^l$. In fact, \mathcal{M}_c is *freely* generated by $\{J^l(z) \mid l \geq 0\}$; the set of iterated Wick products

$$: \partial^{i_1} J^{l_1}(z) \cdots \partial^{i_r} J^{l_r}(z) :,$$

such that $l_1 \leq \cdots \leq l_r$ and $i_a \leq i_b$ if $l_a = l_b$, forms a basis for \mathcal{M}_c .

A weight-homogeneous element $\omega \in \mathcal{M}_c$ is called a *singular vector* if $J^l \circ_k \omega = 0$ for all $k > l \geq 0$. The maximal proper $\hat{\mathcal{D}}$ -submodule \mathcal{I}_c is the vertex algebra ideal generated by all singular vectors $\omega \neq 1$, and the unique irreducible quotient $\mathcal{M}_c/\mathcal{I}_c$ is denoted by $\mathcal{W}_{1+\infty, c}$. The cocycle (3.1) is normalized so that \mathcal{M}_c is reducible if and only if $c \in \mathbb{Z}$. For each integer $n \geq 1$, \mathcal{I}_n is generated by a singular vector of weight $n+1$, and $\mathcal{W}_{1+\infty, n}$ is isomorphic to $\mathcal{W}(\mathfrak{gl}_n)$ with central charge n [FKRW]. In [LI] it was shown that \mathcal{I}_{-n} is generated by a singular vector of weight $(n+1)^2$, and $\mathcal{W}_{1+\infty, -n}$ has a minimal strong generating set consisting of a field in each weight $1, 2, \dots, n^2+2n$. It is known that $\mathcal{W}_{1+\infty, -1}$ is isomorphic to $\mathcal{W}(\mathfrak{gl}_3)$ of central charge -2 [W], but no identification of $\mathcal{W}_{1+\infty, -n}$ with a standard \mathcal{W} -algebra is known for $n > 1$.

4. THE SUPER $\mathcal{W}_{1+\infty}$ ALGEBRA

Following the notation in [CW], we denote by \mathcal{SD} the Lie superalgebra of regular differential operators on the super circle $S^{1|1}$. There is a standard basis for \mathcal{SD} given by

$$t^{k+1}(\partial_t)^l \theta \partial_\theta, \quad t^{k+1}(\partial_t)^l \partial_\theta \theta, \quad t^{k+1}(\partial_t)^l \theta, \quad t^{k+1}(\partial_t)^l \partial_\theta, \quad l \in \mathbb{Z}_{\geq 0}, \quad k \in \mathbb{Z}.$$

Here θ is an odd indeterminate which commutes with t . The odd elements θ and ∂_θ generate a four-dimensional Clifford algebra Cl with relation $\theta \partial_\theta + \partial_\theta \theta = 1$, and $\mathcal{SD} = \mathcal{D} \otimes Cl$.

Let $M(1, 1)$ be the set of 2×2 matrices of the form

$$\begin{pmatrix} \alpha^0 & \alpha^+ \\ \alpha^- & \alpha^1 \end{pmatrix},$$

where $\alpha^a \in \mathbb{C}$ for $a = 0, 1, \pm$. There is a natural \mathbb{Z}_2 -gradation on $M(1, 1)$ where we define $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ to be even, and $M_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $M_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to be odd. The supertrace Str of the above matrix is $\alpha^0 - \alpha^1$. We have an isomorphism $Cl \cong M(1, 1)$ of associative superalgebras given by

$$M_0 \mapsto \partial_\theta \theta, \quad M_1 \mapsto \theta \partial_\theta, \quad M_+ \mapsto \partial_\theta, \quad M_- \mapsto \theta.$$

Therefore we can regard \mathcal{SD} as the superalgebra of 2×2 matrices with coefficients in \mathcal{D} . Let $F(D)$ denote the matrix

$$\begin{pmatrix} f_0(D) & f_+(D) \\ f_-(D) & f_1(D) \end{pmatrix}, \quad D = t\partial_t, \quad f_a(x) \in \mathbb{C}[x],$$

which we regard as an element of \mathcal{SD} . Define a 2-cocycle Ψ on \mathcal{SD} by

$$(4.1) \quad \Psi(t^r F(D), t^s G(D)) = \begin{cases} \sum_{-r \leq j \leq -1} \text{Str}(F(j)G(j+r)) & r = -s \geq 0 \\ 0 & r + s \neq 0. \end{cases}$$

We obtain the corresponding one-dimensional central extension $\widehat{\mathcal{SD}} = \mathcal{SD} \oplus \mathbb{C}C$, with bracket

$$[t^r F(D), t^s G(D)] = t^{r+s}(F(D+s)G(D) - (-1)^{|F||G|}F(D)G(D+r)) + \Psi(t^r F(D), t^s G(D))C.$$

Here $|\cdot|$ denotes the \mathbb{Z}_2 -gradation. We introduce the principal \mathbb{Z} -gradation on $\widehat{\mathcal{SD}}$ by

$$\text{wt}(C) = 0, \quad \text{wt}(t^n f(D)\partial_\theta \theta) = \text{wt}(t^n f(D)\theta \partial_\theta) = n, \quad \text{wt}(t^{n+1} f(D)\partial_\theta) = \text{wt}(t^n f(D)\theta) = n + \frac{1}{2}.$$

This defines the triangular decomposition

$$\widehat{\mathcal{SD}} = \widehat{\mathcal{SD}}_- \oplus \widehat{\mathcal{SD}}_0 \oplus \widehat{\mathcal{SD}}_+, \quad \widehat{\mathcal{SD}}_\pm = \bigoplus_{j \in \mathbb{N}/2} \widehat{\mathcal{SD}}_j.$$

Define $J_n^{a,k} = J_n^k M_a$ for $a = 0, 1, \pm$, and define the parabolic subalgebra $\mathcal{SP} \subset \mathcal{SD}$ to be the Lie algebra spanned by

$$\{J_n^{a,k} | k + n \geq 0, n \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}, a = 0, 1, \pm\}.$$

The cocycle (4.1) vanishes on \mathcal{SP} , so \mathcal{SP} is a subalgebra of $\widehat{\mathcal{SD}}$, and $\widehat{\mathcal{SD}}_0 \oplus \widehat{\mathcal{SD}}_+ \subset \widehat{\mathcal{SP}}$, where $\widehat{\mathcal{SP}} = \mathcal{SP} \oplus \mathbb{C}C$.

Given $c \in \mathbb{C}$, let \mathbf{C}_c denote the one-dimensional $\widehat{\mathcal{SP}}$ -module on which C acts by $c \cdot \text{id}$ and $J_n^{a,k}$ acts by zero. The induced $\widehat{\mathcal{SD}}$ -module

$$\mathcal{M}_c(\widehat{\mathcal{SD}}) = U(\widehat{\mathcal{SD}}) \otimes_{U(\widehat{\mathcal{SP}})} \mathbf{C}_c,$$

is known as the *vacuum $\widehat{\mathcal{SD}}$ -module of central charge c* .

Proposition 4.1. *The vacuum modules at central charge c and $-c$ are isomorphic*

$$\mathcal{M}_c(\widehat{\mathcal{SD}}) \cong \mathcal{M}_{-c}(\widehat{\mathcal{SD}}).$$

Proof. We will construct an automorphism of $\widehat{\mathcal{SD}}$ that maps C to $-C$ and hence $\mathcal{M}_c(\widehat{\mathcal{SD}})$ carries also an action of $\widehat{\mathcal{SD}}$ at central charge $-c$, establishing the isomorphism.

Define the map Π on \mathcal{SD} by

$$\Pi(F(D)) = \begin{pmatrix} f_1(D) & f_-(D) \\ f_+(D) & f_0(D) \end{pmatrix},$$

this map respects the graded commutator of 2×2 matrices and $\Pi \circ \Pi$ acts as the identity, hence Π defines an automorphism on \mathcal{SD} . Π does not respect the supertrace but changes sign, hence also the cocycle satisfies

$$\Psi(t^r \Pi(F(D)), t^s \Pi(G(D))) = -\Psi(t^r F(D), t^s G(D)).$$

Defining $\Pi(C) = -C$ extends Π to an automorphism of $\widehat{\mathcal{SD}}$. □

The module $\mathcal{M}_c(\widehat{\mathcal{SD}})$ possesses a vertex superalgebra structure, and is freely generated by fields

$$\begin{aligned} J^{0,k}(z) &= \sum_{n \in \mathbb{Z}} J_n^{0,k} z^{-n-k-1}, & J^{1,k}(z) &= \sum_{n \in \mathbb{Z}} J_n^{1,k} z^{-n-k-1}, \\ J^{+,k}(z) &= \sum_{n \in \mathbb{Z}} J_n^{+,k} z^{-n-k-1/2}, & J^{-,k}(z) &= \sum_{n \in \mathbb{Z}} J_n^{-,k} z^{-n-k-3/2}, \end{aligned}$$

for $k \geq 0$. Here $J^{0,k}, J^{1,k}, J^{+,k}, J^{-,k}$ have weights $k+1, k+1, k+1/2, k+3/2$, respectively. The modes $J_n^{a,k}$ represent $\widehat{\mathcal{SD}}$ on $\mathcal{M}_c(\widehat{\mathcal{SD}})$, and we rewrite these fields in the form

$$J^{a,k}(z) = \sum_{k \in \mathbb{Z}} J^{a,k}(n) z^{-n-1}, \quad J^{a,k}(n) = J_{n-k}^{a,k}.$$

Define a filtration

$$(\mathcal{M}_c(\widehat{\mathcal{SD}}))_{(0)} \subset (\mathcal{M}_c(\widehat{\mathcal{SD}}))_{(1)} \subset \dots$$

on $\mathcal{M}_c(\widehat{\mathcal{SD}})$ as follows: for $k \geq 0$, $(\mathcal{M}_c(\widehat{\mathcal{SD}}))_{(2k)}$ is the span of iterated Wick products in the generators $J^{a,k}$ and their derivatives of length at most k , and $(\mathcal{M}_c(\widehat{\mathcal{SD}}))_{(2k+1)} = (\mathcal{M}_c(\widehat{\mathcal{SD}}))_{(2k)}$. In particular, each $J^{a,k}$ and its derivatives have degree 2. Equipped with this filtration, $\mathcal{M}_c(\widehat{\mathcal{SD}})$ lies in the category \mathcal{R} , and $\text{gr}(\mathcal{M}_c(\widehat{\mathcal{SD}}))$ is the polynomial superalgebra $\mathbb{C}[\partial^l J^{a,k} \mid l, k \geq 0]$. Each element $J^{a,k}(m) \in \widehat{\mathcal{SP}}$ for $k, m \geq 0$ gives rise to a derivation of degree zero on $\text{gr}(\mathcal{M}_c(\widehat{\mathcal{SD}}))$, and this action of $\widehat{\mathcal{SP}}$ on $\text{gr}(\mathcal{M}_c(\widehat{\mathcal{SD}}))$ is independent of c .

There are some substructures of $\mathcal{M}_c(\widehat{\mathcal{SD}})$ that will be important to us. First, the fields $J^{0,0}, J^{1,0}, J^{+,0}, J^{-,0}$ satisfy

$$\begin{aligned} (4.2) \quad & J^{0,0}(z)J^{0,0}(w) \sim c(z-w)^{-2}, & J^{1,0}(z)J^{1,0}(w) &\sim -c(z-w)^{-2}, \\ & J^{0,0}(z)J^{-,0}(w) \sim J^{-,0}(w)(z-w)^{-1}, & J^{0,0}(z)J^{+,0}(w) &\sim -J^{+,0}(w)(z-w)^{-1}, \\ & J^{1,0}(z)J^{-,0}(w) \sim -J^{-,0}(w)(z-w)^{-1}, & J^{1,0}(z)J^{+,0}(w) &\sim J^{+,0}(w)(z-w)^{-1}, \\ & & J^{+,0}(z)J^{-,0}(w) &\sim c(z-w)^{-2} - (J^{0,0}(w) + J^{1,0}(w))(z-w)^{-1}, \end{aligned}$$

so they generate a copy of the affine vertex superalgebra associated to $\mathfrak{gl}(1|1)$ at level c . Next, recall that the $N = 2$ superconformal vertex algebra \mathcal{A}_c of central charge c is

generated by fields F, L, G^\pm , where L is a Virasoro element of central charge c , F is an even primary of weight one, and G^\pm are odd primaries of weight $\frac{3}{2}$. These fields satisfy

$$(4.3) \quad \begin{aligned} F(z)F(w) &\sim \frac{c}{3}(z-w)^{-1}, & G^\pm(z)G^\pm(w) &\sim 0, \\ F(z)G^\pm(w) &\sim \pm G^\pm(w)(z-w)^{-1}, \\ G^+(z)G^-(w) &\sim \frac{c}{3}(z-w)^{-3} + F(w)(z-w)^{-2} + (L(w) + \frac{1}{2}\partial F(w))(z-w)^{-1}. \end{aligned}$$

We have a vertex algebra homomorphism $\mathcal{A}_c \rightarrow \mathcal{M}_c(\widehat{\mathcal{SD}})$ given by

$$(4.4) \quad F \mapsto \frac{2}{3}J^{0,0} - \frac{1}{3}J^{1,0}, \quad L \mapsto J^{0,1} + J^{1,1} - \frac{2}{3}\partial J^{0,0} - \frac{1}{6}\partial J^{1,1}, \quad G^+ \mapsto J^{-,0}, \quad G^- \mapsto -J^{+,1} + \frac{1}{3}\partial J^{+,0}.$$

Finally, $\{J^{0,k} \mid k \geq 0\}$ and $\{J^{1,k} \mid k \geq 0\}$ generate copies of \mathcal{M}_c and \mathcal{M}_{-c} inside $\mathcal{M}_c(\widehat{\mathcal{SD}})$, respectively.

Lemma 4.2. \mathcal{M}_c and \mathcal{M}_{-c} form a Howe pair, i.e., a pair of mutual commutants, inside $\mathcal{M}_c(\widehat{\mathcal{SD}})$.

Proof. We show that $\text{Com}(\mathcal{M}_c, \mathcal{M}_c(\widehat{\mathcal{SD}})) = \mathcal{M}_{-c}$; the proof that $\text{Com}(\mathcal{M}_{-c}, \mathcal{M}_c(\widehat{\mathcal{SD}})) = \mathcal{M}_c$ is the same. Let $\omega \in \text{Com}(\mathcal{M}_c, \mathcal{M}_c(\widehat{\mathcal{SD}}))$, and write ω as a sum of monomials

$$(4.5) \quad \partial^{a_1} J^{0,i_1} \dots \partial^{a_r} J^{0,i_r} \partial^{b_1} J^{1,j_1} \dots \partial^{b_s} J^{1,j_s} \partial^{c_1} J^{+,k_1} \dots \partial^{c_t} J^{+,k_t} \partial^{d_1} J^{-,l_1} \dots \partial^{d_u} J^{-,l_u}.$$

Since ω commutes with $J^{0,0}$, we have $t = u$ for each such term. Suppose that $u > 0$ for some such monomial, and let l be the maximal integer such that $J^{-,l}$ appears. Since $J^{0,1} \circ_1 J^{-,l} = J^{-,l+1}$, we would have $J^{0,1} \circ_1 \omega \neq 0$, so we conclude that $u = 0$. Therefore $\omega \in \mathcal{M}_c \otimes \mathcal{M}_{-c}$, and since the center of \mathcal{M}_c is trivial, we conclude that $\omega \in \mathcal{M}_{-c}$. \square

Lemma 4.3. For each $c \in \mathbb{C}$, the sets

$$S = \{J^{0,0}, J^{1,0}, J^{+,0}, J^{-,0}, J^{0,1}\}, \quad T = \{J^{0,0}, J^{1,0}, J^{+,0}, J^{-,0}, J^{+,1}, J^{-,1}\}$$

both generate $\mathcal{M}(\widehat{\mathcal{SD}})_c$ as a vertex algebra.

Proof. Let $\langle S \rangle$ and $\langle T \rangle$ denote the vertex subalgebras of $\mathcal{M}(\widehat{\mathcal{SD}})_c$ generated by S and T , respectively. Note that $J^{+,0} \circ_0 J^{0,1} = J^{+,1}$ and $J^{-,0} \circ_0 J^{0,1} = -J^{-,1}$, so $J^{+,1}$ and $J^{-,1}$ both lie in $\langle S \rangle$. Next, $J^{+,0} \circ_0 J^{-,1} = -J^{0,1} - J^{1,1}$, which shows that $J^{1,1} \in \langle S \rangle$. So far, $J^{a,k} \in \langle S \rangle$ for $a = 0, 1, \pm$ and $k = 0, 1$. Next, we have

$$\begin{aligned} J^{0,1} \circ_0 J^{-,1} &= J^{-,2}, & J^{1,1} \circ_0 J^{+,1} &= J^{+,2}, \\ J^{-,2} \circ_2 J^{+,2} - (J^{-,1} \circ_1 J^{+,2}) &= -3J^{1,2}, & J^{-,2} \circ_2 J^{+,2} + 2(J^{-,1} \circ_1 J^{+,2}) &= -6J^{0,2}. \end{aligned}$$

This shows that $J^{a,k} \in \langle S \rangle$ for $a = 0, 1, \pm$ and $k \leq 2$.

For $k \geq 1$, we have

$$J^{0,2} \circ_1 J^{0,k-1} = (k+1)J^{0,k} - 2\partial J^{0,k-1}, \quad J^{0,1} \circ_0 J^{0,k} = \partial J^{0,k}.$$

It follows that $\alpha \circ_1 J^{0,k-1} = (k+1)J^{0,k}$, where $\alpha = J^{0,2} - 2\partial J^{0,1}$. Since $\alpha \in \langle S \rangle$, it follows by induction that $J^{0,k} \in \langle S \rangle$ for all k . Next, we have

$$J^{+,0} \circ_0 J^{0,k} = J^{+,k}, \quad J^{-,0} \circ_0 J^{0,k} = -J^{-,k},$$

so $J^{+,k}$ and $J^{-,k}$ lie in $\langle S \rangle$ for all k . Finally, $J^{+,0} \circ_0 J^{-,k} = -J^{0,k} - J^{1,k}$, which shows that $J^{1,k}$ lies in $\langle S \rangle$ for all k . This shows that $\mathcal{M}(\widehat{\mathcal{SD}})_c = \langle S \rangle$.

To prove that $\mathcal{M}(\widehat{\mathcal{SD}})_c = \langle T \rangle$, it is enough to show that $J^{0,1} \in \langle T \rangle$. First, we have

$$(J^{-,1} \circ_0 J^{+,1}) \circ_1 J^{+,1} = -4J^{+,2} + 2\partial J^{+,1},$$

which implies that $J^{+,2} \in \langle T \rangle$. Finally, we have

$$J^{-,0} \circ_1 J^{+,2} = -2J^{0,1},$$

which shows that $J^{0,1} \in \langle T \rangle$. □

Lemma 4.3 shows that $\mathcal{M}_c(\widehat{\mathcal{SD}})$ is a finitely generated vertex algebra. However, $\mathcal{M}_c(\widehat{\mathcal{SD}})$ is not *strongly* generated by any finite set of vertex operators. This follows from the fact that $\text{gr}(\mathcal{M}_c(\widehat{\mathcal{SD}}))$ is the polynomial superalgebra with generators $\partial^l J^{a,k}$ for $k, l \geq 0$ and $a = 0, 1, \pm$, which implies that there are no nontrivial normally ordered polynomial relations in $\mathcal{M}_c(\widehat{\mathcal{SD}})$. A weight-homogeneous element $\omega \in \mathcal{M}_c(\widehat{\mathcal{SD}})$ is called a *singular vector* if ω is annihilated by the operators

$$J^{0,k} \circ_m, \quad J^{1,k} \circ_m, \quad J^{-,k} \circ_m, \quad J^{+,k} \circ_r, \quad m > k, \quad r \geq k.$$

The maximal proper $\widehat{\mathcal{SD}}$ -submodule \mathcal{SI}_c is the ideal generated by all singular vectors $\omega \neq 1$, and the super $\mathcal{W}_{1+\infty}$ algebra $\mathcal{V}_c(\widehat{\mathcal{SD}})$ is the unique irreducible quotient $\mathcal{M}_c(\widehat{\mathcal{SD}})/\mathcal{SI}_c$. We denote the projection $\mathcal{M}(\widehat{\mathcal{SD}})_c \rightarrow \mathcal{V}_c(\widehat{\mathcal{SD}})$ by π_c , and we write

$$(4.6) \quad j^{a,k} = \pi_c(J^{a,k}), \quad k \geq 0.$$

Clearly $\mathcal{V}_c(\widehat{\mathcal{SD}})$ is generated as a vertex algebra by the corresponding sets

$$\{j^{0,0}, j^{1,0}, j^{+,0}, j^{-,0}, j^{0,1}\}, \quad \{j^{0,0}, j^{1,0}, j^{+,0}, j^{-,0}, j^{+,1}, J^{-,1}\},$$

but there may now be normally ordered polynomial relations among $\{j^{a,k} | k \geq 0\}$ and their derivatives. The actions of $V_c(\mathfrak{gl}(1|1))$ and the $N = 2$ superconformal algebra \mathcal{A}_c on $\mathcal{M}_c(\widehat{\mathcal{SD}})$ descend to actions on $\mathcal{V}_c(\widehat{\mathcal{SD}})$ given by the same formulas, where $J^{a,k}$ is replaced by $j^{a,k}$. Likewise, $\{j^{0,k} | k \geq 0\}$ and $\{j^{1,k} | k \geq 0\}$ generate copies of $\mathcal{W}_{1+\infty,c}$ and $\mathcal{W}_{1+\infty,-c}$ inside $\mathcal{V}_c(\widehat{\mathcal{SD}})$, respectively, and these subalgebras form a Howe pair.

5. THE CASE OF POSITIVE INTEGRAL CENTRAL CHARGE

For $n \in \mathbb{Z}$, $\mathcal{M}_n(\widehat{\mathcal{SD}})$ is reducible and $\mathcal{V}_n(\widehat{\mathcal{SD}})$ has a nontrivial structure. For $n = 0$, $J^{+,0}$ is a singular vector so $V_0(\widehat{\mathcal{SD}}) \cong \mathbb{C}$, and since $\mathcal{V}_n(\widehat{\mathcal{SD}}) \cong \mathcal{V}_{-n}(\widehat{\mathcal{SD}})$ it suffices to consider the case $n \geq 1$. The starting point of our study is a free field realization of $\mathcal{V}_n(\widehat{\mathcal{SD}})$ as the GL_n -invariant subalgebra of the $bc\beta\gamma$ -system \mathcal{F} of rank n [AFMO]. This indicates that the structure of $\mathcal{V}_n(\widehat{\mathcal{SD}})$ is deeply connected to classical invariant theory.

Given a vector space V of dimension n , the $\beta\gamma$ -system $\mathcal{S} = \mathcal{S}(V)$, or algebra of chiral differential operators on V , was introduced in [FMS]. It is the unique even vertex algebra with generators $\beta^x, \gamma^{x'}$ for $x \in V, x' \in V^*$, which satisfy the OPE relations

$$(5.1) \quad \begin{aligned} \beta^x(z)\gamma^{x'}(w) &\sim \langle x', x \rangle (z-w)^{-1}, & \gamma^{x'}(z)\beta^x(w) &\sim -\langle x', x \rangle (z-w)^{-1}, \\ \beta^x(z)\beta^y(w) &\sim 0, & \gamma^{x'}(z)\gamma^{y'}(w) &\sim 0. \end{aligned}$$

There is a one-parameter family of conformal structures

$$(5.2) \quad L_\lambda^S = \lambda \sum_{i=1}^n : \beta^{x_i} \partial \gamma^{x'_i} : + (\lambda - 1) \sum_{i=1}^n : \partial \beta^{x_i} \gamma^{x'_i} :$$

of central charge $n(12\lambda^2 - 12\lambda + 2)$, under which β^x and $\gamma^{x'}$ are primary of conformal weights λ and $1 - \lambda$, respectively. Here $\{x_1, \dots, x_n\}$ is a basis for V and $\{x'_1, \dots, x'_n\}$ is the dual basis for V^* .

Similarly, the bc -system $\mathcal{E} = \mathcal{E}(V)$ is the unique vertex superalgebra with odd generators $b^x, c^{x'}$ for $x \in V, x' \in V^*$, which satisfy the OPE relations

$$(5.3) \quad \begin{aligned} b^x(z)c^{x'}(w) &\sim \langle x', x \rangle (z - w)^{-1}, & c^{x'}(z)b^x(w) &\sim \langle x', x \rangle (z - w)^{-1}, \\ b^x(z)b^y(w) &\sim 0, & c^{x'}(z)c^{y'}(w) &\sim 0. \end{aligned}$$

There is a similar family of conformal structures

$$(5.4) \quad L_\lambda^\mathcal{E} = (1 - \lambda) \sum_{i=1}^n : \partial b^{x_i} c^{x'_i} : - \lambda \sum_{i=1}^n : b^{x_i} \partial c^{x'_i} :$$

of central charge $n(-12\lambda^2 + 12\lambda - 2)$, under which b^x and $c^{x'}$ are primary of conformal weights λ and $1 - \lambda$, respectively. The $bc\beta\gamma$ system \mathcal{F} is just $\mathcal{E} \otimes S$, and we will assign \mathcal{F} the conformal structure

$$L^\mathcal{F} = L_{5/6}^S + L_{1/3}^\mathcal{E},$$

under which $\beta^x, \gamma^{x'}, b^x, c^{x'}$ have weights $5/6, 1/6, 1/3, 2/3$, respectively.

\mathcal{F} admits a good increasing filtration

$$(5.5) \quad \mathcal{F}_{(0)} \subset \mathcal{F}_{(1)} \subset \dots, \quad \mathcal{F} = \bigcup_{k \geq 0} \mathcal{F}_{(k)},$$

where $\mathcal{F}_{(k)}$ is spanned by iterated Wick products of the generators $b^x, c^{x'}, \beta^x, \gamma^{x'}$ and their derivatives, of length at most k . This filtration is GL_n -invariant, and we have an isomorphism of supercommutative rings

$$(5.6) \quad \text{gr}(\mathcal{F}) \cong \text{Sym}\left(\bigoplus_{k \geq 0} (V_k \oplus V_k^*)\right) \otimes \bigwedge\left(\bigoplus_{k \geq 0} (U_k \oplus U_k^*)\right).$$

Here V_k, U_k are copies of V , and V_k^*, U_k^* are copies of V^* . The generators of $\text{gr}(\mathcal{F})$ are $\beta_k^x, \gamma_k^{x'}, b_k^x$, and $c_k^{x'}$, which correspond to the vertex operators $\partial^k \beta^x, \partial^k \gamma^{x'}, \partial^k b^x$, and $\partial^k c^{x'}$, respectively for $k \geq 0$.

Theorem 5.1. (Awata-Fukuma-Matsuo-Otake) *There is an isomorphism $\mathcal{V}_n(\widehat{SD}) \rightarrow \mathcal{F}^{GL_n}$ given by*

$$(5.7) \quad \begin{aligned} j^{0,k} &\mapsto - \sum_{i=1}^n : b^{x_i} \partial^k c^{x'_i} :, & j^{1,k} &\mapsto \sum_{i=1}^n : \beta^{x_i} \partial^k \gamma^{x'_i} :, \\ j^{+,k} &\mapsto - \sum_{i=1}^n : b^{x_i} \partial^k \gamma^{x'_i} :, & j^{-,k} &\mapsto \sum_{i=1}^n : \beta^{x_i} \partial^k c^{x'_i} :. \end{aligned}$$

The Virasoro element L of $\mathcal{V}_n(\widehat{\mathcal{SD}})$ is given by

$$(5.8) \quad L = j^{0,1} + j^{1,1} - \frac{2}{3}\partial j^{0,0} - \frac{1}{6}\partial j^{1,0}.$$

Clearly L maps to $L^{\mathcal{F}}$, and the above map is a morphism in the category \mathcal{R} . The identification $\mathcal{V}_n(\widehat{\mathcal{SD}}) \cong \mathcal{F}^{GL_n}$ suggests an alternative strong generating set for $\mathcal{V}_n(\widehat{\mathcal{SD}})$ coming from classical invariant theory. Since GL_n preserves the filtration on \mathcal{F} , we have

$$(5.9) \quad \text{gr}(\mathcal{V}_n(\widehat{\mathcal{SD}})) \cong \text{gr}(\mathcal{F}^{GL_n}) \cong \text{gr}(\mathcal{F})^{GL_n}.$$

The generators and relations for $\text{gr}(\mathcal{F})^{GL_n}$ are given by Weyl's first and second fundamental theorems of invariant theory for the standard representation of GL_n [We]. This theorem was originally stated for the GL_n -invariants in the symmetric algebra, but the following is an easy generalization to the case of odd as well as even variables.

Theorem 5.2. (Weyl) For $k \geq 0$, let V_k and U_k be the copies of the standard GL_n -module \mathbb{C}^n with basis $x_{i,k}$ and $y_{i,k}$, for $i = 1, \dots, n$, respectively. Let V_k^* and U_k^* be the copies of V^* with basis $x'_{i,k}$ and $y'_{i,k}$, respectively. The invariant ring

$$R = \left(\left(\text{Sym} \bigoplus_{k \geq 0} (V_k \oplus V_k^*) \right) \otimes \left(\bigwedge \bigoplus_{k \geq 0} (U_k \oplus U_k^*) \right) \right)^{GL_n}$$

is generated by the quadratics

$$(5.10) \quad \begin{aligned} q_{a,b}^0 &= \sum_{i=1}^n y_{i,a} y'_{i,b}, & q_{a,b}^1 &= \sum_{i=1}^n x_{i,a} x'_{i,b}, \\ q_{a,b}^+ &= \sum_{i=1}^n y_{i,a} x'_{i,b}, & q_{a,b}^- &= \sum_{i=1}^n x_{i,a} y'_{i,b}. \end{aligned}$$

Let $Q_{k,l}^0, Q_{k,l}^1$ be even indeterminates and let $Q_{k,l}^+, Q_{k,l}^-$ be odd indeterminates for $k, l \geq 0$. The kernel I_n of the homomorphism

$$(5.11) \quad \mathbb{C}[Q_{k,l}^a] \rightarrow R, \quad Q_{k,l}^a \mapsto q_{k,l}^a,$$

is generated by homogeneous polynomials $d_{I,J}$ of degree $n+1$ in the variables $Q_{k,l}^a$. Here $I = (i_0, \dots, i_n)$ and $J = (j_0, \dots, j_n)$ are lists of nonnegative integers, where i_r corresponds to either V_{i_r} or U_{i_r} , and j_s corresponds to either $V_{j_s}^*$ or $U_{j_s}^*$. We call indices i_r and j_s bosonic if they correspond to V_{i_r} and $V_{j_s}^*$, and fermionic if they correspond to U_{i_r} and $U_{j_s}^*$, respectively. Bosonic indices appearing in either I or J must be distinct, but fermionic indices can be repeated. Finally, $d_{I,J}$ is uniquely characterized by the condition that it changes sign if bosonic indices in either I or J are permuted, and remains unchanged if fermionic indices are permuted. If all indices are bosonic,

$$(5.12) \quad d_{I,J} = \det \begin{bmatrix} Q_{i_0,j_0}^1 & \cdots & Q_{i_0,j_n}^1 \\ \vdots & & \vdots \\ Q_{i_n,j_0}^1 & \cdots & Q_{i_n,j_n}^1 \end{bmatrix}.$$

Under the identification (5.9), the generators $q_{k,l}^a$ correspond to strong generators

$$(5.13) \quad \begin{aligned} \omega_{k,l}^0 &= \sum_{i=1}^n : \partial^k b^{x_i} \partial^l c^{x'_i} :, & \omega_{k,l}^1 &= \sum_{i=1}^n : \partial^k \beta^{x_i} \partial^l \gamma^{x'_i} :, \\ \omega_{k,l}^+ &= \sum_{i=1}^n : \partial^k b^{x_i} \partial^l \gamma^{x'_i} :, & \omega_{k,l}^- &= \sum_{i=1}^n : \partial^k \beta^{x_i} \partial^l c^{x'_i} : \end{aligned}$$

of \mathcal{V}_{-n} , satisfying $\phi_2(\omega_{a,b}) = q_{a,b}$. In this notation, we have

$$(5.14) \quad j^{0,k} = -\omega_{0,k}^0, \quad j^{1,k} = \omega_{0,k}^1, \quad j^{+,k} = -\omega_{0,k}^+, \quad j^{-,k} = \omega_{0,k}^-, \quad k \geq 0.$$

For each $m \geq 0$, let A_m^a denote the vector space with basis $\{\omega_{k,l}^a \mid k+l=m\}$. We have $\dim(A_m^a) = m+1$, and $\dim(A_m^a / \partial(A_{m-1}^a)) = 1$. Hence A_m^a has a decomposition

$$(5.15) \quad A_m^a = \partial(A_{m-1}^a) \oplus \langle j^{a,m} \rangle,$$

where $\langle j^{a,m} \rangle$ is the linear span of $j^{a,m}$. Clearly $\{\partial^l j^{0,m} \mid 0 \leq l \leq m\}$ is a basis of A_m , so for $k+l=m$, $\omega_{k,l}^a \in A_m$ can be expressed uniquely in the form

$$(5.16) \quad \omega_{k,l}^a = \sum_{i=0}^m \lambda_i \partial^i j^{a,m-i},$$

for constants λ_i . Hence $\{\partial^k j^{a,m} \mid k, m \geq 0\}$ and $\{\omega_{k,m}^a \mid k, m \geq 0\}$ are related by a linear change of variables. Using (5.16), we can define an alternative strong generating set $\{\Omega_{k,l}^a \mid k, l \geq 0\}$ for $\mathcal{M}_n(\widehat{\mathcal{SD}})$ by the same formula: for $k+l=m$,

$$\Omega_{k,l}^a = \sum_{i=0}^m \lambda_i \partial^i J^{a,m-i}.$$

Clearly $\pi_n(\Omega_{k,l}^a) = \omega_{k,l}^a$.

6. THE STRUCTURE OF THE IDEAL \mathcal{SI}_n

Recall that the projection $\pi_n : \mathcal{M}_n(\widehat{\mathcal{SD}}) \rightarrow \mathcal{V}_n(\widehat{\mathcal{SD}})$ with kernel \mathcal{SI}_n is a morphism in the category \mathcal{R} . Under the identifications

$$\text{gr}(\mathcal{M}_n(\widehat{\mathcal{SD}})) \cong \mathbb{C}[Q_{k,l}^a], \quad \text{gr}(\mathcal{V}_n(\widehat{\mathcal{SD}})) \cong \mathbb{C}[q_{k,l}^a]/I_n,$$

$\text{gr}(\pi_n)$ is just the quotient map (5.11).

Lemma 6.1. *For each classical relation $d_{I,J}$ there exists a unique vertex operator*

$$(6.1) \quad D_{I,J} \in (\mathcal{M}_n(\widehat{\mathcal{SD}}))_{(2n+2)} \cap \mathcal{SI}_n$$

satisfying

$$(6.2) \quad \phi_{2n+2}(D_{I,J}) = d_{I,J}.$$

These elements generate \mathcal{SI}_n as a vertex algebra ideal.

Proof. Clearly π_n maps each filtered piece $(\mathcal{M}_n(\widehat{\mathcal{SD}}))_{(k)}$ onto $(\mathcal{V}_n(\widehat{\mathcal{SD}}))_{(k)}$, so the hypotheses of Lemma 2.2 are satisfied. Since $I_n = \text{Ker}(\text{gr}(\pi_n))$ is generated by $\{d_{I,J}\}$, we can find $D_{I,J} \in (\mathcal{M}_n(\widehat{\mathcal{SD}}))_{(2n+2)} \cap \mathcal{SI}_n$ satisfying $\phi_{2n+2}(D_{I,J}) = d_{I,J}$, such that $\{D_{I,J}\}$ generates

\mathcal{SI}_n . If $D'_{I,J}$ also satisfies (6.2), we would have $D_{I,J} - D'_{I,J} \in (\mathcal{M}_n(\widehat{\mathcal{SD}}))_{(2n)} \cap \mathcal{SI}_n$. Since there are no relations in $\mathcal{V}_n(\widehat{\mathcal{SD}})$ of degree less than $2n + 2$, we have $D_{I,J} - D'_{I,J} = 0$. \square

Recall the generators $b_j^{x_i}, c_j^{x'_i}, \beta_j^{x_i}, \gamma_j^{x'_i}$ of $\text{gr}(\mathcal{F})$ corresponding to $\partial^j b^{x_i}, \partial^j c^{x'_i}, \partial^j \beta^{x_i}, \partial^j \gamma^{x'_i}$. Let $W \subset \text{gr}(\mathcal{F})$ be the vector space with basis $\{b_j^{x_i}, c_j^{x'_i}, \beta_j^{x_i}, \gamma_j^{x'_i} \mid j \geq 0\}$, and for each $m \geq 0$, let W_m be the subspace with basis $\{b_j^{x_i}, c_j^{x'_i}, \beta_j^{x_i}, \gamma_j^{x'_i} \mid 0 \leq j \leq m\}$. Let $\phi : W \rightarrow W$ be a linear map of weight $w \geq 1$, such that

$$(6.3) \quad \phi(b_j^{x_i}) = \lambda_j^b b_{j+w}^{x_i}, \quad \phi(c_j^{x'_i}) = \mu_j^c c_{j+w}^{x'_i}, \quad \phi(\beta_j^{x_i}) = \lambda_j^\beta \beta_{j+w}^{x_i}, \quad \phi(\gamma_j^{x'_i}) = \mu_j^\gamma \gamma_{j+w}^{x'_i}$$

for constants $\lambda_j^b, \mu_j^c, \lambda_j^\beta, \mu_j^\gamma \in \mathbb{C}$ which are independent of i . For example, the restrictions of $j^{0,k}(k-w)$ and $j^{1,k}(k-w)$ to W is such a map for $k > w$.

Lemma 6.2. *Fix $w \geq 1$ and $m \geq 0$, and let ϕ be a linear map satisfying (6.3). Then the restriction $\phi|_{W_m}$ can be expressed uniquely as a linear combination of the operators*

$$\{j^{0,k}(k-w)|_{W_m}, \quad j^{1,k}(k-w)|_{W_m} \mid 0 \leq k \leq 2m+1\}.$$

Proof. The argument is the same as the proof of Lemma 6 of [LII]. \square

Lemma 6.3. *Fix $w \geq 1$ and $m \geq 0$, and let ϕ be a linear map satisfying*

$$(6.4) \quad \phi(b_j^{x_i}) = \lambda_j \beta_{j+w}^{x_i}, \quad \phi(c_j^{x'_i}) = 0, \quad \phi(\beta_j^{x_i}) = 0, \quad \phi(\gamma_j^{x'_i}) = 0.$$

for constants $\lambda_j \in \mathbb{C}$ which are independent of i . Then the restriction $\phi|_{W_m}$ can be expressed as a linear combination of $j^{+,k}(k-w)$ for $0 \leq k \leq 2m+1$.

Similarly, let ψ be a linear map satisfying

$$(6.5) \quad \phi(b_j^{x_i}) = 0, \quad \phi(c_j^{x'_i}) = \mu_j \gamma_{j+w}^{x'_i}, \quad \phi(\beta_j^{x_i}) = 0, \quad \phi(\gamma_j^{x'_i}) = 0.$$

for constants $\mu_j \in \mathbb{C}$. Then the restriction $\psi|_{W_m}$ can be expressed as a linear combination of the operators $j^{-,k}(k-w)|_{W_m}$ for $0 \leq k \leq 2m+1$.

Proof. This is easy to extract from Lemma 6.2 using the $\mathfrak{gl}(1|1)$ structure. \square

Let $\langle D_{I,J} \rangle$ denote the vector space with basis $\{D_{I,J}\}$ where I, J are as in Theorem 5.2. We have $\langle D_{I,J} \rangle = (\mathcal{M}_n(\widehat{\mathcal{SD}}))_{(2n+2)} \cap \mathcal{SI}_n$, and clearly $\langle D_{I,J} \rangle$ is a module over the Lie algebra $\widehat{\mathcal{SP}} \subset \widehat{\mathcal{SD}}$ generated by $\{J^{a,k}(m) \mid m, k \geq 0\}$, since $\widehat{\mathcal{SP}}$ preserves both the filtration on $\mathcal{M}_n(\widehat{\mathcal{SD}})$ and the ideal \mathcal{SI}_n . The action of $\widehat{\mathcal{SP}}$ on $\langle D_{I,J} \rangle$ is by “weighted derivation” in the following sense. Given $I = (i_0, \dots, i_n)$, $J = (j_0, \dots, j_n)$ and given $\phi \in \widehat{\mathcal{SP}}$ satisfying (6.3), we have

$$(6.6) \quad \phi(D_{I,J}) = \sum_{r=0}^n \lambda_{i_r} D_{I^r, J} + \mu_{j_r} D_{I, J^r},$$

for lists $I^r = (i_0, \dots, i_r + w, \dots, i_n)$ and $J^r = (j_0, \dots, j_r + w, \dots, j_n)$. Here $\lambda_{i_r} = \lambda_{i_r}^b$ if i_r is fermionic, and $\lambda_{i_r} = \lambda_{i_r}^\beta$ if bosonic. Moreover, $i_r + w$ has the same parity as i_r , i.e., it is bosonic (respectively fermionic) if and only if i_r is. Similarly, $\mu_{j_r} = \mu_{j_r}^c$ if j_r is fermionic, and $\mu_{j_r} = \mu_{j_r}^\gamma$ if j_r is bosonic, and the parity of μ_{j_r} is preserved. The odd operators $\phi \in \widehat{\mathcal{SP}}$

given by Lemma 6.3 have a similar derivation property except that they reverse the parity of the entries i_r and j_r .

For each $n \geq 1$, there are four distinguished element in $\langle D_{I,J} \rangle$, which correspond to $I = (0, \dots, 0) = J$. Define D_+ to be the element where all entries of I are fermionic, and one entry J is bosonic. Similarly, define D_- to be the element where one entry of I is bosonic and all entries of J are fermionic. Finally, define D_0 to be the element where all entries in both I and J are fermionic, and define D_1 to be the element where one entry of I and one entry of J are bosonic. Clearly D_0, D_1, D_+, D_- have weights $n+1, n+1, n+1/2$, and $n+3/2$, respectively. It is clear that D_+ is the unique element of \mathcal{ST}_n of minimal weight $n+1/2$, and hence is a singular vector in $\mathcal{M}_n(\widehat{\mathcal{SD}})$.

Theorem 6.4. D_+ generates \mathcal{ST}_n as a vertex algebra ideal.

Proof. Since \mathcal{ST}_n is generated by $\langle D_{I,J} \rangle$ as a vertex algebra ideal, it suffices to show that $\langle D_{I,J} \rangle$ is generated by D_+ as a module over $\widehat{\mathcal{SP}}$. Let \mathcal{ST}'_n denote the ideal in $\mathcal{M}_n(\widehat{\mathcal{SD}})$ generated by D_+ , and let $\langle D_{I,J} \rangle^{(m)}$ denote the subspace spanned by elements $D_{I,J}$ with $|I| + |J| = m$. We will prove by induction on m that $\langle D_{I,J} \rangle^{(m)} \subset \mathcal{ST}'_n$.

First we need to show that $\langle D_{I,J} \rangle^{(0)} \subset \mathcal{ST}'_n$, i.e., D_0, D_1 , and D_- lie in \mathcal{ST}'_n . Note that $J^{-,0} \circ_0 D_+ = D_0 + (n+1)D_1$, so $D_0 + (n+1)D_1$ lies in \mathcal{ST}'_n . By Lemma 6.2, we can find $\phi \in \widehat{\mathcal{SP}}$ such that

$$\phi(\beta_0^{x_i}) = \beta_1^{x_i}, \quad \phi(\beta_r^{x'_i}) = 0, \quad \text{for } r > 0, \quad \phi(\gamma_s^{x'_i}) = 0, \quad \phi(c_s^{x'_i}) = 0, \quad \phi(b_s^{x_i}) = 0, \quad s \geq 0.$$

We have $\phi(D_0) = 0$ and $\phi(D_1) = D_{I,J}$ where $I = (1, 0, \dots, 0)$ and $J = (0, \dots, 0)$. Moreover, the entry 1 in I is bosonic, and all other entries of I are fermionic, and J contains one bosonic entry and $n-1$ fermionic entries. It follows that $\phi(D_0 + (n+1)D_1) = (n+1)D_{I,J}$, so $D_{I,J} \in \mathcal{ST}'_n$. Next, note that $j^{1,1}(1)(\beta_1^{x_i}) = 2\beta_0^{x_i}$, so $J^{1,1} \circ_2 (D_{I,J}) = 2D_1$. This shows that $D_1 \in \mathcal{ST}'_n$, so $D_0 \in \mathcal{ST}'_n$ as well. Finally, $J^{-,0} \circ_0 (D_0) = D_-$, so D_- also lies in \mathcal{ST}'_n .

For $m > 0$, we assume inductively that $\langle D_{I,J} \rangle^{(r)}$ lies in \mathcal{ST}'_n for $0 \leq r < m$. Fix $I = (i_0, \dots, i_n)$ and $J = (j_0, \dots, j_n)$ with $|I| + |J| = m$.

Case 1: $I = (0, \dots, 0)$, and j_0, \dots, j_n are all fermionic. Since $m > 0$, at least one of the j_k 's is nonzero. Let J' be obtained from J by replacing j_k with 0. By Lemma 6.2, we can find $\phi \in \widehat{\mathcal{SP}}$ with the property that

$$\phi(c_0^{x'_i}) = c_{j_k}^{x'_i}, \quad \phi(c_r^{x'_i}) = 0 \quad \text{for } r > 0, \quad \phi(\gamma_s^{x'_i}) = 0, \quad \phi(b_s^{x_i}) = 0, \quad \phi(\beta_s^{x_i}) = 0, \quad s \geq 0.$$

Then $\phi(D_{I,J'}) = \lambda D_{I,J}$ where λ is a nonzero constant depending on the number of indices appearing in J which are zero. Since $D_{I,J'} \in \mathcal{ST}'_n[m - j_k]$, we have $D_{I,J} \in \mathcal{ST}'_n$.

Case 2: $I = (0, \dots, 0)$, and for some $0 \leq r < n$, j_0, \dots, j_r are fermionic and j_{r+1}, \dots, j_n are bosonic. If one of the fermionic entries $j_k \neq 0$ for $0 \leq k \leq r$, we proceed as in Case 1. If $j_0 = \dots = j_r$, there exists $j_k > 0$ for some $k = r+1, \dots, n$. Let J' be obtained from J by replacing the bosonic entry j_k with the fermionic entry 0. Then $D_{I,J'} \in \langle D_{I,J} \rangle^{(m-j_k)} \subset \mathcal{ST}'_n$. Using Lemma 6.3, we can find $\phi \in \widehat{\mathcal{SP}}$ such that

$$\phi(c_0^{x'_i}) = \gamma_{j_k}^{x'_i}, \quad \phi(c_r^{x'_i}) = 0 \quad \text{for } r > 0, \quad \phi(\gamma_s^{x'_i}) = 0, \quad \phi(b_s^{x_i}) = 0, \quad \phi(\beta_s^{x_i}) = 0, \quad s \geq 0.$$

It follows that, up to a nonzero constant, $\phi(D_{I,J'}) = D_{I,J}$, so $D_{I,J} \in \mathcal{ST}'_n$.

Case 3: $I \neq (0, \dots, 0)$. This is the same as Cases 1 and 2 with the roles of I and J reversed. \square

7. A MINIMAL STRONG FINITE GENERATING SET FOR $\mathcal{V}_n(\widehat{\mathcal{SD}})$

Recall that $\{j^{0,k} \mid k \geq 0\}$ generate a copy of $\mathcal{W}_{1+\infty,n}$ inside $\mathcal{V}_n(\widehat{\mathcal{SD}})$. It is well known [FKRW] that the relation D_0 above is a singular vector for the action of the Lie subalgebra $\widehat{\mathcal{P}} \subset \widehat{\mathcal{SP}}$, and is of the form

$$J^{0,n} - P(J^{0,0}, \dots, J^{0,n-1}),$$

where P is a normally ordered polynomial in $J^{0,0}, \dots, J^{0,n-1}$ and their derivatives. Applying the projection $\pi_n : \mathcal{M}_n(\widehat{\mathcal{SD}}) \rightarrow \mathcal{V}_n(\widehat{\mathcal{SD}})$ yields a decoupling relation

$$j^{0,n} = P(j^{0,0}, \dots, j^{0,n-1}).$$

This relation is responsible for the isomorphism $\mathcal{W}_{1+\infty,n} \cong \mathcal{W}(\mathfrak{gl}_n)$. In fact, by applying the operators $j^{0,2} \circ_1$ repeatedly, it is easy to construct higher decoupling relations

$$j^{0,m} = Q_m(j^0, j^1, \dots, j^{n-1}),$$

for all $m > n$. In particular, $\{j^{0,k} \mid 0 \leq k < n\}$ strongly generate $\mathcal{W}_{1+\infty,n}$. There are no nontrivial normally ordered polynomial relations among these generators and their derivatives, so they *freely* generate $\mathcal{W}_{1+\infty,n}$.

Theorem 7.1. *The set $\{j^{0,k}, j^{1,k}, j^{+,k}, j^{-,k} \mid k = 0, 1, \dots, n-1\}$ is a minimal strong generating set for $\mathcal{V}_n(\widehat{\mathcal{SD}})$ as a vertex algebra.*

Proof. Via the inclusion $\mathcal{W}_{1+\infty,n} \rightarrow \mathcal{V}_n(\widehat{\mathcal{SD}})$ we obtain decoupling relation

$$(7.1) \quad j^{0,m} = P_m(j^{0,0}, j^{0,1}, \dots, j^{0,n-1}), \quad m \geq n.$$

We shall find all the decoupling relations by acting on this set by the copy of $\mathfrak{gl}(1|1)$ spanned by $j^{a,0}(0)$ for $a = 0, 1, \pm$.

First, acting by $j^{+,0}(0)$ on the relations (7.1) and using the fact that

$$J^{+,0}(0)(\partial^m J^{0,k}) = \partial^m J^{+,k},$$

we get relations

$$(7.2) \quad j^{+,m} = Q_m(j^{0,0}, j^{+,0}, j^{0,1}, j^{+,1}, \dots, j^{0,n-1}, j^{+,n-1}).$$

Similarly, by acting on (7.1) by $j^{-,0}(0)$ and using

$$J^{-,0}(0)(\partial^m J^{0,k}) = -\partial^m J^{-,k},$$

we obtain decoupling relations

$$(7.3) \quad j^{-,m} = R_m(j^{0,0}, j^{-,0}, j^{0,1}, j^{-,1}, \dots, j^{0,n-1}, j^{-,n-1}).$$

Finally, acting by $j^{+,0}(0)$ on (7.3) and using

$$J^{+,0}(0)(\partial^m J^{-,k}) = -\partial^m J^{0,k} - \partial^m J^{1,k},$$

we obtain relations

$$(7.4) \quad j^{0,m} + j^{1,m} = S(j^{0,0}, j^{1,0}, j^{+,0}, j^{-,0}, j^{0,1}, j^{1,1}, j^{+,1}, j^{-,1}, \dots, j^{0,n-1}, j^{1,n-1}, j^{+,n-1}, j^{-,n-1}).$$

We can subtract from this the relation (7.1), obtaining

$$(7.5) \quad j^{1,m} = T(j^{0,0}, j^{1,0}, j^{+,0}, j^{-,0}, j^{0,1}, j^{1,1}, j^{+,1}, j^{-,1}, \dots, j^{0,n-1}, j^{1,n-1}, j^{+,n-1}, j^{-,n-1}).$$

The relations (7.1)-(7.3) and (7.5) imply that $\{j^{0,k}, j^{1,k}, j^{+,k}, j^{-,k} \mid k = 0, 1, \dots, n-1\}$ strongly generates $\mathcal{V}_n(\widehat{\mathcal{SD}})$. The fact that this set is *minimal* is a consequence of Weyl's second fundamental theorem of invariant theory for GL_n ; there are no relations of weight less than $n+1/2$. \square

The cases $n = 1$ and $n = 2$. It is immediate from Theorem 7.1 that $\mathcal{V}_1(\widehat{\mathcal{SD}}) \cong V_1(\mathfrak{gl}(1|1))$. In the case $n = 2$, the decoupling relations for $j^{a,2}$ for $a = 0, 1, \pm$ are as follows:

$$\begin{aligned}
j^{0,2} &= -\frac{1}{6} : j^{0,0} j^{0,0} j^{0,0} : -\frac{1}{2} : j^{0,0} \partial j^{0,0} : + : j^{0,0} j^{0,1} : + \partial j^{0,1} - \frac{1}{6} \partial^2 j^{0,0}. \\
j^{+,2} &= -\frac{1}{2} : j^{+,0} j^{0,0} j^{0,0} : -\frac{1}{2} : j^{+,0} \partial j^{0,0} : + : j^{+,1} j^{0,0} : + : j^{+,0} j^{0,1} : . \\
j^{-,2} &= -\frac{1}{2} : j^{-,0} j^{0,0} j^{0,0} : -\frac{1}{2} : j^{-,0} \partial j^{0,0} : - : \partial j^{-,0} j^{0,0} : + : j^{-,1} j^{0,0} : + : j^{-,0} j^{0,1} : - \partial^2 j^{-,0} + 2 \partial j^{-,1}. \\
\\
j^{1,2} &= - : j^{-,0} j^{+,0} j^{0,0} : -\frac{1}{2} : j^{1,0} j^{0,0} j^{0,0} : -\frac{1}{3} : j^{0,0} j^{0,0} j^{0,0} : - : \partial j^{-,0} j^{+,0} : + : j^{-,1} j^{+,0} : \\
(7.6) \quad &+ : j^{-,0} j^{+,1} : - : \partial j^{1,0} j^{0,0} : -\frac{1}{2} : j^{1,0} \partial j^{0,0} : + : j^{1,1} j^{0,0} : + : j^{1,0} j^{0,1} : \\
&- : j^{0,0} \partial j^{0,0} : + : j^{0,1} j^{0,0} : -\frac{5}{6} \partial^2 j^{0,0} + \partial j^{0,1} - \partial^2 j^{1,0} + 2 \partial j^{1,1}.
\end{aligned}$$

Since the original generating set $\{j^{a,k} \mid k \geq 0\}$ closes linearly under OPE, these decoupling relations allow us to write down all nonlinear OPE relations in $\mathcal{V}_2(\widehat{\mathcal{SD}})$ among the strong generating set $\{j^{a,k} \mid k = 0, 1\}$. For example,

$$(7.7) \quad j^{-,1}(z) j^{+,1}(w) \sim 2(z-w)^{-4} + (j^{1,1} - j^{0,1})(w)(z-w)^{-2} + (\partial j^{1,1} - j^{1,2} - j^{0,2})(w)(z-w)^{-1},$$

which yields

$$\begin{aligned}
j^{-,1}(z) j^{+,1}(w) &\sim 2(z-w)^{-4} + (j^{1,1} - j^{0,1})(w)(z-w)^{-2} + \partial j^{1,1}(w)(z-w)^{-1} \\
&- \left(-\frac{1}{6} : j^{0,0} j^{0,0} j^{0,0} : -\frac{1}{2} : j^{0,0} \partial j^{0,0} : + : j^{0,0} j^{0,1} : + \partial j^{0,1} - \frac{1}{6} \partial^2 j^{0,0} \right. \\
(7.8) \quad &- : j^{-,0} j^{+,0} j^{0,0} : -\frac{1}{2} : j^{1,0} j^{0,0} j^{0,0} : -\frac{1}{3} : j^{0,0} j^{0,0} j^{0,0} : - : \partial j^{-,0} j^{+,0} : + : j^{-,1} j^{+,0} : \\
&+ : j^{-,0} j^{+,1} : - : \partial j^{1,0} j^{0,0} : -\frac{1}{2} : j^{1,0} \partial j^{0,0} : + : j^{1,1} j^{0,0} : + : j^{1,0} j^{0,1} : \\
&\left. - : j^{0,0} \partial j^{0,0} : + : j^{0,1} j^{0,0} : -\frac{5}{6} \partial^2 j^{0,0} + \partial j^{0,1} - \partial^2 j^{1,0} + 2 \partial j^{1,1} \right)(w)(z-w)^{-1}.
\end{aligned}$$

Similarly, we have the following additional nonlinear OPEs:

$$\begin{aligned}
(7.9) \quad j^{0,1}(z) j^{-,1}(w) &\sim j^{-,1}(w)(z-w)^{-2} + \left(\frac{1}{2} : j^{-,0} j^{0,0} j^{0,0} : -\frac{1}{2} : j^{-,0} \partial j^{0,0} : - : \partial j^{-,0} j^{0,0} : \right. \\
&\left. + : j^{-,1} j^{0,0} : + : j^{-,0} j^{0,1} : - \partial^2 j^{-,0} + 2 \partial j^{-,1} \right)(w)(z-w)^{-1},
\end{aligned}$$

$$(7.10) \quad \begin{aligned} j^{1,1}(z)j^{-,1}(w) &\sim j^{-,1}(w)(z-w)^{-2} + \left(\partial j^{-,1} - \frac{1}{2} : j^{-,0}j^{0,0}j^{0,0} : + \frac{1}{2} : j^{-,0}\partial j^{0,0} : \right. \\ &\quad \left. + : \partial j^{-,0}j^{0,0} : - : j^{-,1}j^{0,0} : - : j^{-,0}j^{0,1} : + \partial^2 j^{-,0} - 2\partial j^{-,1} \right)(w)(z-w)^{-1}. \end{aligned}$$

$$(7.11) \quad \begin{aligned} j^{0,1}(z)j^{+,1}(w) &\sim j^{+,1}(w)(z-w)^{-2} + \partial j^{+,1}(z-w)^{-1} - \left(-\frac{1}{2} : j^{+,0}j^{0,0}j^{0,0} : \right. \\ &\quad \left. - \frac{1}{2} : j^{+,0}\partial j^{0,0} : + : j^{+,1}j^{0,0} : + : j^{+,0}j^{0,1} : \right)(w)(z-w)^{-1}. \end{aligned}$$

$$(7.12) \quad \begin{aligned} j^{1,1}(z)j^{+,1}(w) &\sim j^{+,1}(w)(z-w)^{-2} + \left(-\frac{1}{2} : j^{+,0}j^{0,0}j^{0,0} : - \frac{1}{2} : j^{+,0}\partial j^{0,0} : \right. \\ &\quad \left. + : j^{+,1}j^{0,0} : + : j^{+,0}j^{0,1} : \right)(w)(z-w)^{-1}. \end{aligned}$$

The remaining nontrivial OPEs in $\mathcal{V}_2(\widehat{\mathcal{SD}})$ are linear in the generators, and are omitted.

8. A DEFORMABLE FAMILY WITH LIMIT $\mathcal{V}_n(\widehat{\mathcal{SD}})$

We will construct a deformable family of vertex algebras $\mathcal{B}_{n,k}$ with the property that $\mathcal{B}_{n,\infty} = \lim_{k \rightarrow \infty} \mathcal{B}_{n,k} = \mathcal{V}_n(\widehat{\mathcal{SD}})$. The key property will be that for generic values of k , $\mathcal{B}_{n,k}$ has a minimal strong generating set consisting of $4n$ fields, and has the same graded character as $\mathcal{V}_n(\widehat{\mathcal{SD}})$.

First, we need to formalize what we mean by a deformable family. Let $K \subset \mathbb{C}$ be a subset which is at most countable, and let F_K denote the \mathbb{C} -algebra of rational functions in a formal variable κ of the form $\frac{p(\kappa)}{q(\kappa)}$ where $\deg(p) \leq \deg(q)$ and the roots of q lie in K . A *deformable family* will be a free F_K -module \mathcal{B} with the structure of a vertex algebra with coefficients in F_K . Vertex algebras over F_K are defined in the same way as ordinary vertex algebras over \mathbb{C} . We assume that \mathcal{B} possesses a $\mathbb{Z}_{\geq 0}$ -grading $\mathcal{B} = \bigoplus_{m \geq 0} \mathcal{B}[m]$ by conformal weight where each $\mathcal{B}[m]$ is free F_K -module of finite rank. For $k \notin K$, we have a vertex algebra

$$\mathcal{B}_k = \mathcal{B}/(\kappa - k),$$

where $(\kappa - k)$ is the ideal generated by $\kappa - k$. Clearly $\dim_{\mathbb{C}}(\mathcal{B}_k[m]) = \text{rank}_{F_K}(\mathcal{B}[m])$ for all $k \notin K$ and $m \geq 0$. We have a vertex algebra $\mathcal{B}_{\infty} = \lim_{\kappa \rightarrow \infty} \mathcal{B}$ with basis $\{\alpha_i \mid i \in I\}$, where $\{\alpha_i \mid i \in I\}$ is any basis of \mathcal{B} over F_K , and $\alpha_i = \lim_{\kappa \rightarrow \infty} a_i$. By construction, $\dim_{\mathbb{C}}(\mathcal{B}_{\infty}[m]) = \text{rank}_{F_K}(\mathcal{B}[m])$ for all $m \geq 0$. The vertex algebra structure on \mathcal{B}_{∞} is defined by

$$(8.1) \quad \alpha_i \circ_n \alpha_j = \lim_{\kappa \rightarrow \infty} a_i \circ_n a_j, \quad i, j \in I, \quad n \in \mathbb{Z}.$$

It is immediate that the F_K -linear map $\phi : \mathcal{B} \rightarrow \mathcal{B}_{\infty}$ sending $a_i \mapsto \alpha_i$ satisfies

$$(8.2) \quad \phi(\omega \circ_n \nu) = \phi(\omega) \circ_n \phi(\nu), \quad \omega, \nu \in \mathcal{B}, \quad n \in \mathbb{Z}.$$

Moreover, all normally ordered polynomial relations $P(\alpha_i)$ among the generators α_i and their derivatives are of the form

$$\lim_{\kappa \rightarrow \infty} \tilde{P}(a_i),$$

where $\tilde{P}(a_i)$ is a normally ordered polynomial relation among the a_i 's and their derivatives, which converges termwise to $P(\alpha_i)$. In other words, if

$$P(\alpha_i) = \sum_j c_j m_j(\alpha_i)$$

where $m_j(\alpha_i)$ is a normally ordered monomial and $c_j \in \mathbb{C}$, there exists a relation

$$\tilde{P}(a_i) = c_j(\kappa) m_j(a_i)$$

where $\lim_{\kappa \rightarrow \infty} c_j(\kappa) = c_j$ and $m_j(a_i)$ is obtained from $m_j(\alpha_i)$ by replacing α_i with a_i .

We are interested in the relationship between strong generating sets for \mathcal{B}_∞ and \mathcal{B} .

Lemma 8.1. *Let \mathcal{B} be a vertex algebra over F_K as above. Let $U = \{\alpha_i \mid i \in I\}$ be a strong generating set for \mathcal{B}_∞ , and let $T = \{a_i \mid i \in I\}$ be the corresponding subset of \mathcal{B} , so that $\phi(a_i) = \alpha_i$. There exists a subset $S \subset \mathbb{C}$ which is at most countable, such that $F_S \otimes_{\mathbb{C}} \mathcal{B}$ is strongly generated by T . Here we have identified T with the set $\{1 \otimes a_i \mid i \in I\} \subset F_S \otimes_{\mathbb{C}} \mathcal{B}$.*

Proof. Without loss of generality, we may assume that U is linearly independent. Complete U to a basis U' for \mathcal{B}_∞ containing finitely many element in each weight, and let T' be the corresponding basis of \mathcal{B} over F_K . Let d be the first weight such that U' contains elements which do not lie in U , and let $\alpha_{1,d}, \dots, \alpha_{r,d}$ be the set of elements of $U' \setminus U$ of weight d . Since U strongly generates \mathcal{B}_∞ , we have decoupling relations in \mathcal{B}_∞ of the form

$$\alpha_{j,d} = P_j(\alpha_i), \quad j = 1, \dots, r.$$

Here P is a normally ordered polynomial in the generators $\{\alpha_i \mid i \in I\}$ and their derivatives. Let $a_{j,d}$ be the corresponding elements of T' . There exist relations

$$a_{j,d} = \tilde{P}_j(a_i, a_{1,d}, \dots, \widehat{a_{j,d}}, \dots, a_{r,d}), \quad j = 1, \dots, r,$$

which converge termwise to $P_j(\alpha_i)$. Here \tilde{P}_j does not depend on $a_{j,d}$ but may depend on $a_{k,d}$ for $k \neq j$. Since each $a_{k,d}$ has weight d and \tilde{P}_j is homogeneous of weight d , \tilde{P}_j depends linearly on $a_{k,d}$. We can therefore rewrite these relations in the form

$$\sum_{k=1}^r b_{jk} a_{k,d} = Q_j(a_i), \quad b_{jk} \in F_K,$$

where $b_{jj} = 1$, $\lim_{\kappa \rightarrow \infty} b_{jk} = 0$ for $j \neq k$, and

$$Q_j(a_i) = \tilde{P}_j(a_i, a_{1,d}, \dots, \widehat{a_{j,d}}, \dots, a_{r,d}) + \sum_{k=1}^{j-1} b_{jk} a_{k,d} + \sum_{k=j+1}^r b_{jk} a_{k,d}.$$

Clearly $\lim_{\kappa \rightarrow \infty} \det[b_{jk}] = 1$, so this matrix is invertible over the field of rational functions in κ . Let S_d denote the set of distinct roots of the numerator of $\det[b_{jk}]$ regarded as a rational function of κ . We can solve this linear system over the ring F_{S_d} , so in $F_{S_d} \otimes_{\mathbb{C}} \mathcal{B}$ we obtain decoupling relations

$$t_{a,d} = \tilde{Q}_j(a_i), \quad j = 1, \dots, r.$$

For each weight $d+1, d+2, \dots$ we repeat this procedure, obtaining finite sets

$$S_d \subset S_{d+1} \subset S_{d+2} \subset \dots$$

and decoupling relations

$$a = P(a_i)$$

in $F_{S_{d+i}} \otimes_{\mathbb{C}} \mathcal{B}$, for each $a \in T' \setminus T$ of weight $d + i$. Letting $S = \bigcup_{i \geq 0} S_{d+i}$, we obtain decoupling relations in $F_S \otimes_{\mathbb{C}} \mathcal{B}$ expressing each $a \in T' \setminus T$ as a normally ordered polynomial in a_1, \dots, a_s and their derivatives. \square

Corollary 8.2. *For $k \notin K \cup S$, the vertex algebra $\mathcal{B}_k = \mathcal{B}/(\kappa - k)$ is strongly generated by the image of T under the map $\mathcal{B} \rightarrow \mathcal{B}_k$.*

Next we consider a special class of deformable families that are well known in the physics literature. Let \mathcal{V} be a vertex algebra equipped a conformal weight grading $\mathcal{V} = \bigoplus_{m \geq 0} \mathcal{V}[m]$ with each $\mathcal{V}[m]$ finite-dimensional. Let \mathfrak{g} be a simple, finite-dimensional Lie algebra. Fix an orthonormal basis ξ_1, \dots, ξ_n for \mathfrak{g} relative to the normalized Killing form, so that the generators X^{ξ_i} of $V_l(\mathfrak{g})$ satisfy

$$X^{\xi_i}(z)X^{\xi_j}(w) \sim l\delta_{i,j}(z-w)^{-2} + X^{[\xi_i, \xi_j]}(w)(z-w)^{-1}.$$

Let $V_l(\mathfrak{g}) \rightarrow \mathcal{V}$ be a vertex algebra homomorphism and assume that the action of \mathfrak{g} on \mathcal{V} integrates to an action of a connected, reductive group G on \mathcal{V} with $\mathfrak{g} = \text{Lie}(G)$, so that \mathcal{V}^G coincides with the joint kernel of the zero modes $X^{\xi_i}(0)$.

It is well-known (see [BFH]) that \mathcal{V}^G admits a deformation as follows. We have the diagonal homomorphism $V_{k+l}(\mathfrak{g}) \rightarrow V_k(\mathfrak{g}) \otimes \mathcal{V}$ sending $\tilde{X}^{\xi} \mapsto \tilde{X}^{\xi} \otimes 1 + 1 \otimes X^{\xi}$. Here \tilde{X}^{ξ_i} and \bar{X}^{ξ_i} are the generators of $V_k(\mathfrak{g})$ and $V_{k+l}(\mathfrak{g})$, respectively. Define

$$\mathcal{B}_k = \text{Com}(V_{k+l}(\mathfrak{g}), V_k(\mathfrak{g}) \otimes \mathcal{V}).$$

There is a linear map $\mathcal{B}_k \rightarrow \mathcal{V}^G$ defined as follows. Each element $\omega \in \mathcal{B}_k$ of weight d can be written uniquely in the form $\omega = \sum_{r=0}^d \omega_r$ where ω_r lies in the space

$$(8.3) \quad (V_k(\mathfrak{g}) \otimes \mathcal{V})^{(r)}$$

spanned by terms of the form $\alpha \otimes \nu$ where $\alpha \in V_k(\mathfrak{g})$ has weight r . Clearly $\omega_0 \in \mathcal{V}^G$ so we have a well-defined linear map

$$(8.4) \quad \phi_k : \mathcal{B}_k \rightarrow \mathcal{V}^G, \quad \omega \mapsto \omega_0.$$

Note that ϕ_k is not a vertex algebra homomorphism for any k .

Lemma 8.3. *ϕ_k is surjective for $k \neq 0$.*

Proof. Fix $\nu \in \mathcal{V}^G$ of weight d . For $k \neq 0$, we will construct $\omega \in \mathcal{B}_k$ such that $\phi_k(\omega) = \nu$. First, let $\omega_0 = \nu$ and

$$(8.5) \quad \omega_1 = -\frac{1}{k} \sum_{i=1}^n \tilde{X}^{\xi_i} \otimes (X^{\xi_i} \circ_1 \nu).$$

Clearly $\omega_0 + \omega_1$ is G -invariant (equivalently, it is annihilated by $\bar{X}^{\xi_i} \circ_0$ for $i = 1, \dots, n$), and has the property that $\bar{X}^{\xi_i} \circ_1 (\omega_0 + \omega_1)$ lies in $(V_k(\mathfrak{g}) \otimes \mathcal{V})^{(1)}$. Inductively, suppose that ω_{r-1} has been defined so that ω_{r-1} is G -invariant and $\sum_{s=0}^{r-1} (\bar{X}^{\xi_i} \circ_1 \omega_s)$ lies in $(V_k(\mathfrak{g}) \otimes \mathcal{V})^{(r-1)}$. Define

$$(8.6) \quad \omega_r = -\frac{1}{k} \sum_{i=1}^n \left(\tilde{X}^{\xi_i} \otimes \sum_{s=0}^{r-1} (\bar{X}^{\xi_i} \circ_1 \omega_s) \right).$$

Clearly ω_r is G -invariant and $\sum_{s=0}^r (\bar{X}^{\xi_i} \circ_1 \omega_s)$ lies in $(V_k(\mathfrak{g}) \otimes \mathcal{V})^{(r)}$. This process terminates after at most d steps, and $\omega = \sum_{r=0}^d \omega_r$ lies in \mathcal{B}_k since ω is G -invariant and is annihilated by $\tilde{X}^{\xi_i} \circ_1$ for $i = 1, \dots, n$. By definition, $\phi_k(\omega) = \nu$. \square

Lemma 8.4. ϕ_k is injective whenever $V_k(\mathfrak{g})$ is a simple vertex algebra.

Proof. Assume that $V_k(\mathfrak{g})$ is simple. Fix $\omega \in \mathcal{B}_k$, and suppose that $\phi_k(\omega) = 0$. If $\omega \neq 0$, there is a minimal integer $r > 0$ such that $\omega_r \neq 0$. We may express ω_r as a linear combination of terms of the form $\alpha \otimes \nu$ for which the ν 's are linearly independent. Since ω lies in the commutant \mathcal{B}_k , it follows that each of the above α 's must be annihilated by $\tilde{X}^{\xi_i}(m)$ for $i = 1, \dots, n$ and all $m > 0$. Since $\text{wt}(\alpha) = r > 0$, this implies that α generates a nontrivial ideal in $V_k(\mathfrak{g})$, which is a contradiction. \square

Let $K \subset \mathbb{C}$ be the set of values of k such that $V_k(\mathfrak{g})$ is *not* simple. This set is countable and is described explicitly by Theorem 0.2.1 in the paper [GK] by Kac-Gorelik. As above, there exists a vertex algebra \mathcal{B} with coefficients in F_K with the property that $\mathcal{B}/(\kappa - k) = \mathcal{B}_k$ for all $k \notin K$. The generators of \mathcal{B} are the same as the generators of \mathcal{B}_k , where k has been replaced by the formal variable κ , and the OPE relations are the same as well. The maps ϕ_k above give rise to a linear isomorphism $\phi_\kappa : \mathcal{B} \rightarrow F_K \otimes_{\mathbb{C}} \mathcal{V}^G$, which is not a vertex algebra homomorphism.

Corollary 8.5. The induced map $\phi = \lim_{\kappa \rightarrow \infty} \phi_\kappa$ is a vertex algebra isomorphism from $\mathcal{B}_\infty \rightarrow \mathcal{V}^G$.

Proof. It is clear from (8.5) and (8.6) that ϕ is a vertex algebra homomorphism. Since $\dim(\mathcal{B}_\infty[m]) = \dim(\mathcal{B}_k[m]) = \dim(\mathcal{V}^G[m])$ for all $k \notin K$ and all $m \geq 0$, ϕ must be an isomorphism. \square

Corollary 8.6. Let $\{\nu_i \mid i \in I\}$ be a strong generating set for \mathcal{V}^G , and let $\{\omega_i \mid i \in I\}$ be the corresponding subset of \mathcal{B}_k , where $\phi_k(\omega_i) = \nu_i$. Then $\{\omega_i \mid i \in I\}$ strongly generates \mathcal{B}_k for generic values of k .

Proof. This is immediate from Lemma 8.1 and Corollary 8.2. \square

Corollary 8.7. Suppose that $\{\nu_i \mid i \in I\}$ generates \mathcal{V}^G , not necessarily strongly. Then the corresponding subset $\{\omega_i \mid i \in I\}$ generates \mathcal{B}_k for generic values of k .

Proof. This is immediate from the fact that if $\{\nu_i \mid i \in I\}$ generates \mathcal{V}^G , the set

$$\{\nu_{i_1} \circ_{j_1} (\cdots (\nu_{i_{r-1}} \circ_{j_{r-1}} \nu_{i_r}) \cdots) \mid i_1, \dots, i_r \in I, j_1, \dots, j_{r-1} \geq 0\}$$

strongly generates \mathcal{V}^G . \square

Now we specialize this construction to the example where \mathcal{V} is the rank n $bc\beta\gamma$ -system \mathcal{F} , which carries an action of $V_0(\mathfrak{gl}_n)$ which is just the sum of the action of $V_1(\mathfrak{gl}_n)$ on the bc -system \mathcal{E} and $V_{-1}(\mathfrak{gl}_n)$ on the $\beta\gamma$ -system \mathcal{S} . Even though \mathfrak{gl}_n is not semisimple, the proof of Lemma 8.3 is easily modified to handle this case. First, any element $\nu \in \mathcal{F}^{GL_n}$ can be corrected to an element $\omega \in \text{Com}(V_k(\mathfrak{sl}_n), V_k(\mathfrak{gl}_n) \otimes \mathcal{F})$ such that $\phi_k(\omega) = \nu$. We can further correct ω to make it invariant under the Heisenberg algebra corresponding to the central term in \mathfrak{gl}_n without destroying the property $\phi_k(\omega) = \nu$.

We have $\mathcal{F}^{GL_n} \cong \mathcal{V}_n(\widehat{\mathcal{SD}})$, and we obtain a deformable family of vertex algebras

$$\mathcal{B}_{n,k} = \text{Com}(V_k(\mathfrak{gl}_n), V_k(\mathfrak{gl}_n) \otimes \mathcal{F}),$$

such that $\mathcal{B}_{n,\infty} \cong \mathcal{V}_n(\widehat{\mathcal{SD}})$ and $\mathcal{B}_{n,k}$ has the same graded character as $\mathcal{V}_n(\widehat{\mathcal{SD}})$ for generic values of k .

Theorem 8.8. Let U be the strong generating set $\{j^{0,l}, j^{1,l}, j^{+,l}, j^{-,l} \mid l = 0, 1, \dots, n-1\}$ for $\mathcal{V}_n(\widehat{\mathcal{SD}})$ given by Theorem 7.1, and let $T = \{t^{0,l}, t^{1,l}, t^{+,l}, t^{-,l} \mid l = 0, 1, \dots, n-1\}$ be the corresponding subset of $\mathcal{B}_{n,k}$ with $\phi_k(t^{a,l}) = j^{a,l}$. For generic values of k , T is a minimal strong generating set for $\mathcal{B}_{n,k}$.

Proof. By Lemma 8.1 and Corollary 8.2, T strongly generates $\mathcal{B}_{n,k}$ for generic k . If T were not minimal, we would have a decoupling relation expressing $t^{a,l}$ as a normally ordered polynomial in the remaining elements of T and their derivatives, for some $l \leq n-1$. This relation has weight at most $n + 1/2$, and taking the limit as $k \rightarrow \infty$ would give us a nontrivial relation in $\mathcal{V}_n(\widehat{\mathcal{SD}})$ of the same weight. But this is impossible by Weyl's theorem, which implies that there are no relations in \mathcal{F}^{GL_n} of weight less than $n + 1/2$. \square

9. \mathcal{W} -ALGEBRAS OF $\widehat{\mathfrak{gl}}(n|n)$

As mentioned in the introduction, \mathcal{W} -algebras can often be realized in various ways. In this section, we find a family of \mathcal{W} -algebras $\mathcal{W}_{n,k}$ associated to a certain simple and purely odd root system of $\widehat{\mathfrak{gl}}(n|n)$. We will see that $\mathcal{V}_n(\widehat{\mathcal{SD}}) = \lim_{k \rightarrow \infty} \mathcal{W}_{n,k}$. In the case $n = 2$ we write down all nontrivial OPE relations in $\mathcal{W}_{2,k}$ explicitly.

Definition 9.1. Let $X = \{E_{ij} \mid 1 \leq i, j \leq 2n\}$ be the basis of a \mathbb{Z}_2 -graded \mathbb{C} -vector space with gradation given by

$$(9.1) \quad |E_{ij}| = \begin{cases} 0 & \text{for } 1 \leq i, j \leq n \text{ or } n+1 \leq i, j \leq 2n \\ 1 & \text{for } 1 \leq i \leq n, n+1 \leq j \leq 2n \text{ or } 1 \leq j \leq n, n+1 \leq i \leq 2n \end{cases}$$

Then

$$(9.2) \quad [E_{ij}, E_{kl}] = \delta_{j,k} E_{il} - (-1)^{|E_{ij}||E_{kl}|} \delta_{i,l} E_{kj}$$

provides the \mathbb{C} -span of X with the structure of a Lie superalgebra, namely $\widehat{\mathfrak{gl}}(n|n)$. A consistent, graded symmetric, invariant and non-degenerate bilinear form is given by

$$(9.3) \quad B(E_{ij}, E_{kl}) = \delta_{j,k} \delta_{i,l} \times \begin{cases} 1 & \text{for } 1 \leq i \leq n \\ -1 & \text{for } n+1 \leq i \leq 2n \end{cases}$$

A Cartan subalgebra has a basis given by the E_{ii} . A root system is given by α_{ij} for $1 \leq i \neq j \leq 2n$ with $\alpha_{ij}(E_{kk}) = \delta_{ik} - \delta_{jk}$. The parity of a root is defined as $|\alpha_{ij}| = |E_{ij}|$. The distinguished system of positive simple roots is given by α_{ii+1} , where only α_{nn+1} is an odd root. We are interested in a system of positive simple and purely odd roots. Such a system is given by $\{\alpha_i = \alpha_{i,n+i}, \beta_i = \alpha_{n+i,i+1}\}$. Define $2n$ bosonic fields ϕ_i^\pm , $1 \leq i \leq n$, and $2n$ fermionic fields ψ_i^\pm , with operator products

$$(9.4) \quad \phi_i^\pm(z) \phi_j^\pm(w) \sim \pm k \delta_{i,j} \ln(z-w), \quad \psi_i^\pm(z) \psi_j^\pm(w) \sim \pm \frac{k \delta_{i,j}}{(z-w)}.$$

and all others are regular. The complex number k will be called level. We define the even and odd Cartan subalgebra valued fields

$$(9.5) \quad \phi = \sum_{i=1}^n \phi_i^+ E_{ii} + \phi_i^- E_{n+i,n+i}, \quad \psi = \sum_{i=1}^n \psi_i^+ E_{ii} + \psi_i^- E_{n+i,n+i}.$$

Finally, we are ready to define the set of screening associated to our purely odd simple root system

$$(9.6) \quad \begin{aligned} Q_{\alpha_i} &= \text{Res}_z (: \alpha_i(\psi(z)) e^{\alpha_i(\phi(z))} :) \\ Q_{\beta_i} &= \frac{1}{k} \text{Res}_z (: \beta_i(\psi(z)) e^{\beta_i(\phi(z))} :) \end{aligned}$$

It is convenient to change basis as follows.

$$(9.7) \quad \begin{aligned} Y_i(z) &= \phi_i^+(z) - \phi_i^-(z) \\ X_i(z) &= \frac{1}{2k} (\phi_i^+ + \phi_i^- + \sum_{j=1}^{i-1} Y_j(z) - \sum_{j=i+1}^n Y_j(z)) \\ b_i(z) &= \psi_i^+(z) - \psi_i^-(z) \\ c_i(z) &= \frac{1}{2k} (\psi_i^+ + \psi_i^- + \sum_{j=1}^{i-1} b_j(z) - \sum_{j=i+1}^n b_j(z)) \end{aligned}$$

in this new basis, the non-regular OPEs are

$$(9.8) \quad Y_i(z) X_j(w) \sim \delta_{i,j} \ln(z-w), \quad b_i(z) c_j(w) \sim \frac{\delta_{i,j}}{(z-w)}.$$

The screening charges read in this basis

$$(9.9) \quad \begin{aligned} Q_{\alpha_i} &= \text{Res}_z (: b_i(z) e^{Y_i(z)} :) \\ Q_{\beta_i} &= \text{Res}_z (: (c_i(z) - c_{i+1}(z)) e^{k(X_i(z) - X_{i+1}(z))} :) \end{aligned}$$

Let M_i be the vertex algebra generated by $\partial Y_i(z), \partial X_i(z), b(z), c(z)$ and let $M = \bigoplus_i M_i$. We have

Lemma 9.2. *Let*

$$\begin{aligned} N_i(z) &= \partial X_i(z) - : b_i(z) c_i(z) :, & E_i(z) &= \partial Y_i(z), \\ \Psi_i^+(z) &= b_i(z), & \Psi_i^-(z) &= \partial c_i(z) - : c_i(z) \partial Y_i(z) : \end{aligned}$$

then the vertex algebra generated by N_i, E_i, Ψ_i^\pm is a homomorphic image of $V_1(\mathfrak{gl}(1|1))$, moreover this algebra is contained in the kernel of Q_{α_i} .

Proof. The non-regular operator products of N_i, E_i, Ψ_i^\pm are

$$(9.10) \quad \begin{aligned} N_i(z) E_i(w) &\sim \frac{1}{(z-w)^2} \\ N_i(z) N_i(w) &\sim \frac{1}{(z-w)^2} \\ N_i(z) \Psi_i^\pm(w) &\sim \frac{\mp \Psi_i^\pm(w)}{(z-w)} \\ \Psi_i^+(z) \Psi_i^-(w) &\sim -\frac{1}{(z-w)^2} - \frac{E_i(w)}{(z-w)} \end{aligned}$$

which coincides with the operator product algebra of $V_1(\mathfrak{gl}(1|1))$. E_i, N_i, Ψ_i^+ are obviously in the kernel of Q_{α_i} . The statement for Ψ_i^- follows from

$$: b_i(z) e^{Y_i(z)} : \Psi_i^-(w) \sim \frac{: e^{Y_i(w)} :}{(z-w)^2}.$$

□

Lemma 9.3. *We have*

$$\begin{aligned} E_i + E_{i+1}, N_i + N_{i+1} - \frac{1}{k} E_i, \Psi_i^\pm + \Psi_{i+1}^\pm &\in \text{Ker}_M(Q_{\beta_i}) \\ : \Psi_i^+ N_i : + : \Psi_{i+1}^+ N_{i+1} : - \frac{1}{k} \partial \Psi_i^+ &\in \text{Ker}_M(Q_{\beta_i}) \\ : N_i \Psi_i^- : + : N_{i+1} \Psi_{i+1}^- : + \frac{1}{k} (: E_{i+1} \Psi_i^- : - : E_i \Psi_{i+1}^- :) - \frac{1}{k} \partial \Psi_i^- &\in \text{Ker}_M(Q_{\beta_i}) \end{aligned}$$

Proof. The first two lines are obvious, while the last one is a lengthy OPE computation. One needs

$$\begin{aligned} : N_i(z) \Psi_i^-(z) : &= : b_i(z) : \partial c_i(z) c_i(z) :: - : c_i(z) : \partial X_i(z) \partial Y_i(z) :: + \\ &+ : \partial c_i(z) \partial X_i(z) : - \partial c_i(z) \partial Y_i(z) : + \frac{1}{2} \partial^2 c_i(z). \end{aligned}$$

□

We define some fields in M

$$\begin{aligned} E(z) &= - \sum_{i=1}^n E_i(z), & N(z) &= - \sum_{i=1}^n N_i(z) + \frac{1}{k} \sum_{i=1}^n (n-i) E_i(z) \\ \Psi^+(z) &= \sum_{i=1}^n \Psi_i^+(z), & \Psi^-(z) &= \sum_{i=1}^n \Psi_i^-(z) \\ (9.11) \quad F^+(z) &= \sum_{i=1}^n : \Psi_i^+(z) N_i(z) : - \frac{1}{k} \sum_{i=1}^n (n-i) \partial \Psi_i^+(z) \\ F^-(z) &= \sum_{i=1}^n : N_i(z) \Psi_i^-(z) : - \frac{1}{k} \sum_{i=1}^n (n-i) \partial \Psi_i^-(z) + \\ &+ \frac{1}{2k} \left(\sum_{1 \leq i < j \leq n} : E_j(z) \Psi_i^-(z) : - \sum_{1 \leq j < i \leq n} : E_j(z) \Psi_i^-(z) : \right) \end{aligned}$$

Theorem 9.4.

$$E, N, \Psi^\pm, F^\pm \in \bigcap_{i=1}^n \text{Ker}_M(Q_{\alpha_i}) \cap \bigcap_{i=1}^{n-1} \text{Ker}_M(Q_{\beta_i})$$

Proof. By Lemma 9.2 we have $E, N, \Psi^\pm, F^\pm \in \bigcap_{i=1}^n \text{Ker}_M(Q_{\alpha_i})$ and Lemma 9.3 implies $E, N, \Psi^\pm, F^\pm \in \bigcap_{i=1}^{n-1} \text{Ker}_M(Q_{\beta_i})$ □

Definition 9.5. *We call the vertex algebra generated by E, N, Ψ^\pm, F^\pm by $\mathcal{W}_{n,k}$. We define $\mathcal{W}_{n,\infty}$ as the limit $k \rightarrow \infty$.*

Theorem 9.6. $\mathcal{V}_n(\widehat{SD}) \cong \mathcal{W}_{n,\infty}$

Proof. We will construct the isomorphism explicitly. Let $\tilde{\beta}_i, \tilde{\gamma}_i, \tilde{b}_i, \tilde{c}_i$ be the generators of a rank n $bc\beta\gamma$ -ghost vertex algebra. Then let $\tilde{\phi}_i^\pm, \tilde{\phi}_i$ be bosonic fields with OPE

$$\tilde{\phi}_i^\pm(z)\tilde{\phi}_j^\pm(w) \sim \pm\delta_{i,j}\ln(z-w), \quad \tilde{\phi}_i(z)\tilde{\phi}_j(w) \sim \delta_{i,j}\ln(z-w).$$

Using the well-known bosonization isomorphism, one obtains

$$\tilde{b}_i(z) =: e^{-\tilde{\phi}_i(z)} :, \quad \tilde{c}_i(z) =: e^{\tilde{\phi}_i(z)} :, \quad \tilde{b}_i(z)\tilde{c}_i(z) := -\partial\tilde{\phi}_i(z)$$

and

$$\tilde{\beta}_i(z) =: e^{-\tilde{\phi}_i^-(z)+\tilde{\phi}_i^+(z)}\partial\tilde{\phi}_i^+(z) :, \quad \tilde{\gamma}_i(z) =: e^{\tilde{\phi}_i^-(z)-\tilde{\phi}_i^+(z)} :, \quad \tilde{\beta}_i(z)\tilde{\gamma}_i(z) := \partial\tilde{\phi}_i^-(z)$$

We define

$$Y_i := \tilde{\phi}_i - \tilde{\phi}_i^-, \quad X_i = \tilde{\phi}_i^- - \tilde{\phi}_i^+ \quad \phi = \tilde{\phi}_- - \tilde{\phi}_i^- + \tilde{\phi}_i^+$$

then the non-zero OPEs of these fields are

$$Y_i(z)X_j(w) \sim \delta_{i,j}\ln(z-w), \quad \phi_i(z)\phi_j(w) \sim \delta_{i,j}\ln(z-w).$$

Finally we use bosonization again to obtain

$$b_i(z) =: e^{-\phi_i(z)} :, \quad c_i(z) =: e^{\phi_i(z)} :, \quad b_i(z)c_i(z) := -\partial\phi_i(z).$$

Under this isomorphism, we get the following identifications

$$\begin{aligned} E_i(z) &= \partial Y_i(z) = \partial\tilde{\phi}_i(z) - \partial\tilde{\phi}_i^-(z) = - : \tilde{b}_i(z)\tilde{c}_i(z) : - : \tilde{\beta}_i(z)\tilde{\gamma}_i(z) : \\ N_i(z) &= \partial Y_i(z) - : b_i(z)c_i(z) : = \partial\tilde{\phi}_i(z) = - : \tilde{b}_i(z)\tilde{c}_i(z) : \\ (9.12) \quad \Psi_i^+(z) &= b_i(z) = : e^{-\phi_i(z)} : = : e^{-\tilde{\phi}_i(z)+\tilde{\phi}_i^-(z)-\tilde{\phi}_i^+(z)} : = : \tilde{b}_i(z)\tilde{\gamma}_i(z) : \\ \Psi_i^-(z) &= \partial c_i(z) - : c_i(z)\partial Y_i(z) : = : e^{\phi_i(z)} : (\partial\phi_i(z) - \partial Y_i(z)) \\ &= : e^{\tilde{\phi}_i(z)-\tilde{\phi}_i^-(z)+\tilde{\phi}_i^+(z)} : \partial\tilde{\phi}_i^+(z) = : \tilde{c}_i(z)\tilde{\beta}_i(z) : \end{aligned}$$

and hence

$$\begin{aligned} E(z) &= \sum_{i=1}^n : \tilde{b}_i(z)\tilde{c}_i(z) : + : \tilde{\beta}_i(z)\tilde{\gamma}_i(z) :, & N(z) &= \sum_{i=1}^n : \tilde{b}_i(z)\tilde{c}_i(z) : \\ \Psi^+(z) &= \sum_{i=1}^n : \tilde{b}_i(z)\tilde{\gamma}_i(z) :, & \Psi^-(z) &= \sum_{i=1}^n : \tilde{c}_i(z)\tilde{\beta}_i(z) : \\ F^+(z) &= \sum_{i=1}^n : \Psi_i^+(z)N_i(z) : = \sum_{i=1}^n : \tilde{b}_i(z)\tilde{\gamma}_i(z) : \tilde{b}_i(z)\tilde{c}_i(z) := - \sum_{i=1}^n : \tilde{b}_i(z)\partial\tilde{\gamma}_i(z) : \\ F^-(z) &= \sum_{i=1}^n : N_i(z)\Psi_i^-(z) : = \sum_{i=1}^n : \tilde{b}_i(z)\tilde{c}_i(z) : \tilde{c}_i(z)\tilde{\beta}_i(z) := - \sum_{i=1}^n : \tilde{\beta}_i(z)\partial\tilde{c}_i(z) : \end{aligned}$$

But these fields are by Lemma 4.3 a generating set of $\mathcal{V}_n(\widehat{\mathcal{SD}})$. □

The operator product algebra of $\mathcal{W}_{2,k}$ can be computed explicitly. For this, we choose a slightly different basis from (9.11). Let $n = 2$, and define

$$G^\pm = F^\pm \pm \left(\frac{1}{2k} - \frac{1}{2} \right) \partial\Psi^\pm.$$

There will be two additional fields, a Virasoro field of central charge zero,

$$T = : E_1(z)N_1(z) : + : E_2(z)N_2(z) : - : \Psi_1^+(z)\Psi_1^-(z) : - : \Psi_2^+(z)\Psi_2^-(z) : + \\ - \frac{1+k}{2k}\partial E_1(z) + \frac{1-k}{2k}\partial E_2(z)$$

and another dimension two field

$$H = -\frac{1}{2}(: N_1(z)N_1(z) : + : N_2(z)N_2(z) : - : E_1(z)N_1(z) : - : E_2(z)N_2(z) : + \\ + : \Psi_1^+(z)\Psi_1^-(z) : - : \Psi_2^+(z)\Psi_2^-(z) :) + \frac{1}{2k}(: E_1(z)N_2(z) : - : E_2(z)N_1(z) : + \\ + : \Psi_1^+(z)\Psi_2^-(z) : - : \Psi_2^+(z)\Psi_1^-(z) : + \frac{1}{k} : E_1(z)E_2(z) : + \partial N_1(z) - \partial N_2(z)) + \\ - \frac{1}{8k^2}((2k^2 + 2k + 1)\partial E_1(z) + (2k^2 - 2k + 1)\partial E_2(z))$$

Then E, N, Ψ^\pm have the operator product algebra of $\widehat{\mathfrak{gl}}(1|1)$ at level two, and G^\pm are Virasoro primaries of dimension two, while H is the *partner* of T ,

$$T(z)H(w) \sim \left(\frac{3}{k^2} - 1\right)\frac{1}{(z-w)^4} + \frac{3}{4k^2}\frac{E(w)}{(z-w)^3} + \frac{2H(w)}{(z-w)^2} + \frac{\partial H(w)}{(z-w)}.$$

In addition the operator products of the dimension two fields with the currents are

$$N(z)H(w) \sim \frac{3}{2k^2}\frac{1}{(z-w)^3} - \left(\frac{1}{4} - \frac{3}{4k^2}\right)\frac{E(w)}{(z-w)^2}, \quad N(z)G^\pm(w) \sim \frac{\pm G^\pm(w)}{(z-w)} \\ \Psi^\pm(z)H(w) \sim -\frac{G^\pm(w)}{(z-w)}, \quad \Psi^\pm(z)G^\mp(w) \sim \frac{N(w)}{(z-w)^2} \pm \frac{T(w)}{(z-w)} \\ E(z)H(w) \sim -\frac{N(w)}{(z-w)^2}, \quad E(z)G^\pm(w) \sim -\frac{\pm \psi^\pm(w)}{(z-w)^2}.$$

Introduce the following normally ordered polynomials in the currents and their derivatives

$$X_0 = \frac{1}{2} \left(2 : HE : - 2 : TE : - 2 : TN : - 2 : G^+\Psi^- : - 2 : G^-\Psi^+ : + : \partial\Psi^-\Psi^+ : + : \partial\Psi^+\Psi^- : \right. \\ \left. + : \partial EN : - 2 : N : \Psi^+\Psi^- : + : N : NE : - : E : \Psi^+\Psi^- : + : N : EE : \right) \\ - \frac{1}{8k^2} \left((1 - 2k^2)\partial^2 E + (3 - 2k^2) : \partial EE : + (1 - k^2) : E : EE : \right) \\ X^+ = \frac{1}{2} \left(: N\partial\Psi^+ : - 2 : H\Psi^+ : - 2 : NG^+ : + : T\Psi^+ : - : EG^+ : - : N : N\Psi^+ : - : N : E\Psi^+ : \right) \\ - \frac{1}{8k^2} \left((2 + 2k^2)\partial^2\Psi^+ - : \partial E\Psi^+ : - (2 + 2k^2) : E\partial\Psi^+ : - (1 - k^2) : E : E\psi^+ : \right) \\ X^- = \frac{1}{2} \left(2 : NG^- : - : N\partial\Psi^- : - 2 : H\Psi^- : + : T\Psi^- : + : EG^- : - : N : N\Psi^- : - : N : E\Psi^- : \right) \\ + \frac{1}{8k^2} \left((2 + 2k^2)\partial^2\Psi^- + 5 : \partial E\Psi^- : - (2 + 2k^2) : E\partial\Psi^- : + (1 - k^2) : E : E\psi^- : \right) \\ X_2 = 3\partial^2 N + (2 + 2k^2)\partial^2 E + 4 : \partial\psi^-\psi^+ : - 4 : \partial\psi^+\psi^- : + 4 : \partial NE : + 4 : \partial EN : + 2 : \partial EE :$$

Then the operator products of the dimension two fields with themselves are

$$\begin{aligned}
H(z)H(w) &\sim -\frac{1}{4k^2} \frac{1}{(z-w)^2} \left(2\partial E(w) + 3\partial N(w) - (2k^2 + 2)T(w) + 4 : N(w)E(w) : + \right. \\
&\quad \left. : E(w)E(w) : - 4 : \psi^+(w)\psi^-(w) : \right) - \frac{1}{8k^2} \frac{X_2(w)}{(z-w)} \\
H(z)G^+(w) &\sim \left(\frac{1}{2} - \frac{3}{4k^2} \right) \frac{\Psi^+(w)}{(z-w)^3} + \left(\frac{1}{4} - \frac{3}{4k^2} \right) \frac{\partial \Psi^+(w)}{(z-w)^2} + \frac{G^+(w) + X^+(w)}{(z-w)} \\
H(z)G^-(w) &\sim \left(\frac{1}{2} - \frac{9}{4k^2} \right) \frac{\Psi^-(w)}{(z-w)^3} + \left(\frac{1}{4} - \frac{3}{4k^2} \right) \frac{\partial \Psi^-(w)}{(z-w)^2} + \frac{G^-(w) + X^-(w)}{(z-w)} \\
G^+(z)G^-(w) &\sim -\left(1 - \frac{3}{k^2} \right) \frac{1}{(z-w)^4} - \left(\frac{1}{2} - \frac{3}{2k^2} \right) \frac{E(w)}{(z-w)^3} - \frac{1}{4} \frac{\partial E(w) - 8H(w)}{(z-w)^2} \\
&\quad + (H(w) + X_0(w))(z-w)^{-1}.
\end{aligned}$$

We see

Proposition 9.7. $\mathcal{W}_{2,k}$ is strongly generated by $E, N, \Psi^\pm, T, H, G^\pm$.

10. THE RELATIONSHIP BETWEEN $\mathcal{B}_{n,k}$ AND $\mathcal{W}_{n,k}$

Recall the algebra $\mathcal{B}_{n,k}$ that we constructed in Section 8, which has a minimal strong generating set consisting of $4n$ fields, for generic values of k . In this section we show that for $n = 2$, $\mathcal{W}_{2,k+2}$ and $\mathcal{B}_{2,k}$ have the same generators and OPE relations. More generally, we conjecture that $\mathcal{W}_{n,k+n}$ is isomorphic to $\mathcal{B}_{n,k}$ for all n and k .

Recall that $V_k(\mathfrak{gl}_n)$ has a strong generating set $\{X^{ij} | 1 \leq i, j \leq n\}$ satisfying

$$(10.1) \quad X^{ij}(z)X^{lm}(w) \sim \frac{k\delta_{j,l}\delta_{i,m}}{(z-w)^2} + \frac{\delta_{j,l}X^{im}(w) - \delta_{i,m}X^{lj}(w)}{(z-w)}.$$

Recall the $bc\beta\gamma$ system $\mathcal{F} = \mathcal{E} \otimes \mathcal{S}$ of rank n . There is a map $V_1(\mathfrak{gl}_n) \rightarrow \mathcal{E}$ sending $X^{ij} \mapsto : c_i b_j :$ and a map $V_{-1}(\mathfrak{gl}_n) \rightarrow \mathcal{S}$ sending $X^{ij} \mapsto - : \gamma_i \beta_j :$. These combine to give us a map $V_0(\mathfrak{gl}_n) \rightarrow \mathcal{F}$ sending $X^{ij} \mapsto c_i b_j : - : \gamma_i \beta_j :$.

A straightforward computation shows that $\mathcal{B}_{n,k} = \text{Com}(V_k(\mathfrak{gl}_n), V_k(\mathfrak{gl}_n) \otimes \mathcal{F})$ contains the following elements:

$$\begin{aligned}
\tilde{\Psi}^- &= \sum_{l=1}^n : c_l \beta_l :, & \tilde{\Psi}^+ &= - \sum_{l=1}^n : \gamma_l b_l :, \\
\tilde{E} &= - \sum_{l=1}^n : c_l b_l : + : \gamma_l \beta_l :, & \tilde{N} &= \sum_{l=1}^n \frac{2}{k} X^{ll} - : c_l b_l : + : \gamma_l \beta_l :, \\
\tilde{F}^- &= \frac{1}{k} \sum_{1 \leq k, l \leq n} : X^{kl} c_l \beta_k : + \sum_{l=1}^n : c_l \partial \beta_l :, & \tilde{F}^+ &= \frac{1}{k} \sum_{1 \leq k, l \leq n} : X^{kl} \gamma_l b_k : + \sum_{l=1}^n : \gamma_l \partial b_l :.
\end{aligned}$$

By Lemma 4.3, the elements of $\mathcal{B}_{n,\infty} = \mathcal{V}_n(\widehat{\mathcal{SD}})$ corresponding to these six elements under $\phi_k : \mathcal{B}_{n,k} \rightarrow \mathcal{V}_n(\widehat{\mathcal{SD}})$, are a generating set for $\mathcal{V}_n(\widehat{\mathcal{SD}})$. By Corollary 8.7, these six fields generate $\mathcal{B}_{n,k}$ for generic values of k .

Theorem 10.1. For generic values of k , $\mathcal{W}_{2,k+2}$ and $\mathcal{B}_{2,k}$ have the same OPE algebra.

Proof. This is a computer computation, where the field identification is given by, with $s = (1 + k)/(4 + 2k)$,

$$(10.2) \quad \begin{aligned} E &\rightarrow \tilde{E}, & N &\rightarrow \tilde{N} - \frac{\tilde{E}}{k}, & \Psi^\pm &\rightarrow \tilde{\Psi}^\pm \\ G^+ &\rightarrow (4s - 1)\tilde{F}^+ + s\partial\tilde{\Psi}^+ + (2s - 1) : \tilde{N}\tilde{\Psi}^+ :, \\ G^- &\rightarrow (4s - 1)\tilde{F}^- - (3s - 1)\partial\tilde{\Psi}^- - (2s - 1) : \tilde{N}\tilde{\Psi}^- :. \end{aligned}$$

□

Remark 10.2. *There exist other realizations of $\mathcal{W}_{2,k}$. Let E_{ij} for $1 \leq i, j \leq 4$ be the basis of $\mathfrak{gl}(2|2)$ of Definition 9.1, and we denote the corresponding fields of $V_k(\mathfrak{gl}(2|2))$ by $E_{ij}(z)$. Then, we find that for level $k = -2$ the fields*

$$\begin{aligned} E' &= -\frac{1}{2}\left(\sum_{i=1}^4 E_{ii}\right), & N' &= \frac{1}{2}(E_{11} + E_{22} - E_{33} - E_{44}), \\ \Psi^{+'} &= \frac{1}{\sqrt{-2}}(E_{13} + E_{24}), & \Psi^{-'} &= \frac{1}{\sqrt{-2}}(E_{31} + E_{42})' \\ G^{+'} &= \frac{1}{\sqrt{-2}}(: E_{12}E_{23} : + : E_{21}E_{14} : + : (E_{11} - E_{22})(E_{13} - E_{24}) :) + \\ &\quad - \frac{1}{2}\partial\Psi^{+'} - \frac{1}{2} : N'\Psi^{+'} : + \frac{1}{4} : E'\Psi^{+'} : \\ G^{-'} &= \frac{1}{\sqrt{-2}}(: E_{12}E_{41} : + : E_{21}E_{32} : + : (E_{11} - E_{22})(E_{31} - E_{42}) :) + \\ &\quad - \frac{1}{2}\partial\Psi^{-'} + \frac{1}{2} : N'\Psi^{-'} : - \frac{1}{4} : E'\Psi^{-'} : \end{aligned}$$

are elements of $\text{Com}(V_0(\mathfrak{sl}(2)), V_{-2}(\mathfrak{gl}(2|2)))$ and satisfy the operator product algebra of $\mathcal{W}_{2,-1}$ where the field identification is given by $X \rightarrow X'$, for $X = E, N, \Psi^\pm, G^\pm$.

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